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ON INEQUALITIES OF SIMPSON'S TYPE FOR CONVEX FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. Fractional calculus and applications have application areas in many different fields such as physics, chemistry, and engineering as well as mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. In this study, we prove an identity involving generalized fractional integrals by using differentiable functions. By utilizing this identity, we obtain several Simpson's type inequalities for the functions whose derivatives in absolute value are convex. Finally, we present some new results as the special cases of our main results.

1. Introduction

Simpson's rules are well-known ways for the numerical integration and numerical estimation of definite integrals. This method is known as developed by Thomas Simpson's (1710–1761). However, Johannes Kepler used the same approximation about 100 years ago, so that this method is also known as Kepler's rule. Simpson's rule includes the three-point Newton-Cotes quadrature rule, so estimation based on three steps quadratic kernel is sometimes called as Newton type results.

(1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\kappa_1}^{\kappa_2} \vartheta(\chi)\mathrm{d}\chi \approx \frac{\kappa_2-\kappa_1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right].$$

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(2) Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule).

$$\int_{\kappa_1}^{\kappa_2} \vartheta(\chi)\mathrm{d}\chi \approx \frac{\kappa_2-\kappa_1}{8}\left[\vartheta(\kappa_1)+3\vartheta\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\vartheta\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\vartheta(\kappa_2)\right].$$

There are a large number of estimations related to these quadrature rules in the literature, one of them is the following estimation known as Simpson's inequality:

Theorem 1. *Suppose that $\vartheta:[\kappa_1,\kappa_2]\to\mathbb{R}$ is a four times continuously differentiable mapping on (κ_1,κ_2) and $\left\|\vartheta^{(4)}\right\|_\infty=\sup_{\chi\in(\kappa_1,\kappa_2)}\left|\vartheta^{(4)}(\chi)\right|<\infty$. Then, one has the inequality*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{\vartheta(\kappa_1)+\vartheta(\kappa_2)}{2}+2\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\vartheta(\chi)d\chi\right|\\&\leq\frac{1}{2880}\left\|\vartheta^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.\end{aligned}$$

In recent years, many authors have focused on Simpson's type inequalities for various classes of functions. Specifically, some mathematicians have worked on Simpson's and Newton's type results for convex mappings, because convexity theory is an effective and powerful method for solving a large number of problems which arise within different branches of pure and applied mathematics. For example, Dragomir et al. [16] presented new Simpson's type results and their applications to quadrature formulas in numerical integration. What is more, some inequalities of Simpson's type for s-convex functions are deduced by Alomari et al. in [6]. Afterwards, Sarikaya et al. observed the variants of Simpson's type inequalities based on convexity in [42]. In [34] and [35], the authors provided some Newton's type inequalities for harmonic convex and p-harmonic convex functions. Additionally, new Newton's type inequalities for functions whose local fractional derivatives are generalized convex are given by Iftikhar et al. in [25]. For more recent developments, one can consult [2–5, 7, 11–15, 17, 18, 23, 36, 47].

2. Generalized Fractional Integrals

Fractional calculus and applications have application areas in many different fields such as physics, chemistry and engineering as well as mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. Many real dynamical systems are better characterized by using non-integer order dynamic models based on fractional computation. While integer orders are a model that is not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined enables us to obtain more realistic approaches. This subject has been studied by many scientists in terms of its widespread use [20, 21, 27, 30, 31, 37, 40, 44]. One of the most important applications of the fractional Integrals is the Hermite-Hadamard integral inequality (see, [1, 22, 26, 38, 39, 41]).

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [41].

Let's define a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{\kappa_1+}I_{\varphi}\vartheta(\chi) = \int_{\kappa_1}^{\chi} \frac{\varphi(\chi - \tau)}{\chi - \tau}\vartheta(\tau)d\tau, \quad \chi > \kappa_1, \tag{1}$$

$$_{\kappa_2-}I_{\varphi}\vartheta(\chi) = \int_{\chi}^{\kappa_2} \frac{\varphi(\tau - \chi)}{\tau - \chi}\vartheta(\tau)d\tau, \quad \chi < \kappa_2. \tag{2}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (1) and (2) are mentioned below.

i) If we take $\varphi(\tau) = \tau$, the operator (1) and (2) reduce to the Riemann integral as follows:

$$I_{\kappa_1+}\vartheta(\chi) = \int_{\kappa_1}^{\chi} \vartheta(\tau)d\tau, \quad \chi > \kappa_1,$$

$$I_{\kappa_2-}\vartheta(\chi) = \int_{\chi}^{\kappa_2} \vartheta(\tau)d\tau, \quad \chi < \kappa_2.$$

ii) Let us consider $\varphi(\tau) = \frac{\tau^{\alpha}}{\Gamma(\alpha)}$, $\alpha > 0$. Then, the operator (1) and (2) reduce to the Riemann-Liouville fractional integral as follows:

$$J_{\kappa_1+}^{\alpha}\vartheta(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\chi} (\chi - \tau)^{\alpha - 1} \vartheta(\tau)d\tau, \quad \chi > \kappa_1,$$

$$J_{\kappa_2-}^{\alpha}\vartheta(\chi) = \frac{1}{\Gamma(\alpha)} \int_{\chi}^{\kappa_2} (\tau - \chi)^{\alpha - 1} \vartheta(\tau)d\tau, \quad \chi < \kappa_2.$$

iii) For $\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$, $\alpha, k > 0$, the operator (1) and (2) reduce to the k-Riemann-Liouville fractional integral as follows:

$$J_{\kappa_1+,k}^{\alpha}\vartheta(\chi) = \frac{1}{k\Gamma_k(\alpha)} \int_{\kappa_1}^{\chi} (\chi - \tau)^{\frac{\alpha}{k} - 1} \vartheta(\tau)d\tau, \quad \chi > \kappa_1,$$

$$J_{\kappa_2-,k}^{\alpha}\vartheta(\chi)=\frac{1}{k\Gamma_k(\alpha)}\int_{\chi}^{\kappa_2}(\tau-\chi)^{\frac{\alpha}{k}-1}\vartheta(\tau)d\tau,\quad \chi<\kappa_2.$$

Here,

$$\Gamma_k(\alpha)=\int_0^{\infty}\tau^{\alpha-1}e^{-\frac{\tau^k}{k}}d\tau,\quad \mathcal{R}(\alpha)>0$$

and

$$\Gamma_k(\alpha)=k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right),\quad \mathcal{R}(\alpha)>0;k>0$$

are given by Mubeen and Habibullah in [33].

In the literature, there are several papers on inequalities for generalized fractional integrals. For more information and unexplained subjects, we refer the reader to [8–10, 19, 24, 28, 29, 32, 46, 48] and the references therein.

3. Simpson's Type Inequalities for Generalized Fractional Integrals

Throughout this study for brevity, we define

$$\eta_1(\chi,\tau)=\int_0^{\tau}\frac{\varphi((\kappa_2-\chi)u)}{u}du,\quad \nu_1(\chi,\tau)=\int_0^{\tau}\frac{\varphi((\chi-\kappa_1)u)}{u}du.$$

Particularly, if we choose $\chi=\frac{\kappa_1+\kappa_2}{2}$, then we have

$$\eta_1\left(\frac{\kappa_1+\kappa_2}{2},\tau\right)=\nu_1\left(\frac{\kappa_1+\kappa_2}{2},\tau\right)=\Upsilon_1(\tau)=\int_0^{\tau}\frac{\varphi\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)u\right)}{u}du.$$

Lemma 1. *Let $\vartheta:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an absolutely continuous mapping (κ_1,κ_2) such that $\vartheta'\in L_1([\kappa_1,\kappa_2])$. Then, the following equality holds:*

$$\begin{aligned}&\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\\ =\;&\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\int_0^1(\eta_1(\chi,1)-3\eta_1(\chi,\tau))\vartheta'(\tau\chi+(1-\tau)\kappa_2)d\tau\\ &-\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\int_0^1(\nu_1(\chi,1)-3\nu_1(\chi,\tau))\vartheta'((1-\tau)\kappa_1+\tau\chi)d\tau.\end{aligned}$$

Proof. By using integration by parts, we have

$$\begin{aligned}H_1&=\int_0^1(\eta_1(\chi,1)-3\eta_1(\chi,\tau))\vartheta'(\tau\chi+(1-\tau)\kappa_2)d\tau \qquad (3)\\ &=\frac{1}{\kappa_2-\chi}\eta_1(\chi,1)\left[2\vartheta(\chi)+\vartheta(\kappa_2)\right]\end{aligned}$$

$$+\frac{3}{\chi-\kappa_2}\int_0^1 \vartheta\left(\tau\chi+(1-\tau)\kappa_2\right)\frac{\varphi\left((\kappa_2-\chi)\tau\right)}{\tau}d\tau$$
$$=\frac{1}{\kappa_2-\chi}\eta_1(\chi,1)\left[2\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{3}{\kappa_2-\chi}\int_\chi^{\kappa_2}\frac{\vartheta(u)\varphi(\kappa_2-u)}{\kappa_2-u}du$$
$$=\frac{\eta_1(\chi,1)}{\kappa_2-\chi}\left[2\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{3}{\kappa_2-\chi}\ {}_{\chi+}I_\varphi\vartheta(\kappa_2).$$

Similarly, we obtain

$$H_2=\int_0^1\left(\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right)\vartheta'\left(\tau\chi+(1-\tau)\kappa_1\right)d\tau \tag{4}$$
$$=\frac{\nu_1(\chi,1)}{\chi-\kappa_1}\left[-2\vartheta(\chi)-\vartheta(\kappa_1)\right]+\frac{3}{\chi-\kappa_1}\ {}_{\chi-}I_\varphi\vartheta(\kappa_1).$$

From (3) and (4), we get

$$\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}H_1-\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}H_2$$
$$=\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_\varphi\vartheta(\kappa_2)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_\varphi\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right].$$

This ends the proof of Lemma 1. □

Corollary 1. *Under assumptions of Lemma 1 with* $\chi=\frac{\kappa_1+\kappa_2}{2}$, *we obtain the equality*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]$$
$$-\frac{1}{2\Upsilon_1(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\vartheta(\kappa_2)+{}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\vartheta(\kappa_1)\right]$$
$$=\frac{\kappa_2-\kappa_1}{12\eta_1(\chi,1)}\int_0^1\left(\Upsilon_1(1)-3\Upsilon_1(\tau)\right)$$
$$\times\left[\vartheta'\left(\frac{\tau}{2}\kappa_1+\frac{2-\tau}{2}\kappa_2\right)-\vartheta'\left(\frac{2-\tau}{2}\kappa_1+\frac{\tau}{2}\kappa_2\right)\right]d\tau.$$

Corollary 2. *In Lemma 1, if we choose* $\varphi(\tau)=\tau$ *for all* $\tau\in[\kappa_1,\kappa_2]$, *then we obtain the equality*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{1}{\kappa_2-\chi}\int\limits_{\chi}^{\kappa_2}\vartheta(\tau)d\tau+\frac{1}{\chi-\kappa_1}\int\limits_{\kappa_1}^{\chi}\vartheta(\tau)d\tau\right]$$

$$=\frac{\kappa_2-\chi}{6}\int\limits_0^1(1-3\tau)\,\vartheta'(\tau\chi+(1-\tau)\kappa_2)\,d\tau$$

$$-\frac{\chi-\kappa_1}{6}\int\limits_0^1(1-3\tau)\,\vartheta'((1-\tau)\kappa_1+\tau\chi)\,d\tau.$$

Corollary 3. *In Lemma 1, let us consider* $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)},\alpha>0$ *for all* $\tau\in[\kappa_1,\kappa_2]$. *Then, we get the equality*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{J^{\alpha}_{\chi+}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\alpha}}+\frac{J^{\alpha}_{\chi-}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\alpha}}\right]$$

$$=\frac{\kappa_2-\chi}{6}\int\limits_0^1(1-3\tau^{\alpha})\,\vartheta'(\tau\chi+(1-\tau)\kappa_2)\,d\tau$$

$$-\frac{\chi-\kappa_1}{6}\int\limits_0^1(1-3\tau^{\alpha})\,\vartheta'((1-\tau)\kappa_1+\tau\chi)\,d\tau.$$

Corollary 4. *In Lemma 1, if we assign* $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},k,\alpha>0$ *for all* $\tau\in[\kappa_1,\kappa_2]$, *then we have the equality*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma_k(\alpha+k)}{2}\left[\frac{J^{\alpha}_{\chi+,k}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\frac{\alpha}{k}}}+\frac{J^{\alpha}_{\chi-,k}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\frac{\alpha}{k}}}\right]$$

$$=\frac{\kappa_2-\chi}{6}\int\limits_0^1\left(1-3\tau^{\frac{\alpha}{k}}\right)\vartheta'(\tau\chi+(1-\tau)\kappa_2)\,d\tau$$

$$-\frac{\chi-\kappa_1}{6}\int\limits_0^1\left(1-3\tau^{\frac{\alpha}{k}}\right)\vartheta'((1-\tau)\kappa_1+\tau\chi)\,d\tau.$$

Remark 1. *If we set* $\chi=\frac{\kappa_1+\kappa_2}{2}$ *in Corollaries 2, 3 and 4, then we obtain the following identities*

$$\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\vartheta(\tau)\,d\tau$$

$$= \frac{\kappa_2-\kappa_1}{12}\left[\int_0^1 (1-3\tau)\,\vartheta'\left(\frac{\tau}{2}\kappa_1+\frac{2-\tau}{2}\kappa_2\right)d\tau - \int_0^1 (1-3\tau)\,\vartheta'\left(\frac{2-\tau}{2}\kappa_1+\frac{\tau}{2}\kappa_2\right)d\tau\right],$$

$$\begin{aligned}&\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right] \\ &\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\vartheta(\kappa_1)\right] \\ &= \frac{\kappa_2-\kappa_1}{12}\left[\int_0^1 (1-3\tau^{\alpha})\,\vartheta'\left(\frac{\tau}{2}\kappa_1+\frac{2-\tau}{2}\kappa_2\right)d\tau \right. \\ &\quad \left. - \int_0^1 (1-3\tau^{\alpha})\,\vartheta'\left(\frac{2-\tau}{2}\kappa_1+\frac{\tau}{2}\kappa_2\right)d\tau\right],\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right] \\ &\quad - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\vartheta(\kappa_1)\right] \\ &= \frac{\kappa_2-\kappa_1}{12}\left[\int_0^1 \left(1-3\tau^{\frac{\alpha}{k}}\right)\vartheta'\left(\frac{\tau}{2}\kappa_1+\frac{2-\tau}{2}\kappa_2\right)d\tau \right. \\ &\quad \left. - \int_0^1 \left(1-3\tau^{\frac{\alpha}{k}}\right)\vartheta'\left(\frac{2-\tau}{2}\kappa_1+\frac{\tau}{2}\kappa_2\right)d\tau\right],\end{aligned}$$

respectively.

Theorem 2. *Assume that the assumptions of Lemma 1 hold. Assume also that the mapping $|\vartheta'|$ is convex on $[\kappa_1,\kappa_2]$. Then, we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\right| \\ &\le \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left[\Xi_1\left|\vartheta'(\chi)\right|+\Xi_2\left|\vartheta'(\kappa_2)\right|\right]+\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left[\Xi_3\left|\vartheta'(\kappa_1)\right|+\Xi_4\left|\vartheta'(\chi)\right|\right],\end{aligned}$$

where

$$\begin{cases} \Xi_1 = \int_0^1 \tau \left|\eta_1(\chi,1) - 3\eta_1(\chi,\tau)\right| d\tau, \\ \Xi_2 = \int_0^1 (1-\tau) \left|\eta_1(\chi,1) - 3\eta_1(\chi,\tau)\right| d\tau, \\ \Xi_3 = \int_0^1 (1-\tau) \left|\nu_1(\chi,1) - 3\nu_1(\chi,\tau)\right| d\tau, \\ \Xi_4 = \int_0^1 \tau \left|\nu_1(\chi,1) - 3\nu_1(\chi,\tau)\right| d\tau. \end{cases} \tag{5}$$

Proof. By taking modulus in Lemma 1, we obtain

$$\begin{aligned} &\left|\frac{1}{6}\left[\vartheta(\kappa_1) + 4\vartheta(\chi) + \vartheta(\kappa_2)\right] - \frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)} + \frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\right| \\ &\le \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\int_0^1 \left|\eta_1(\chi,1) - 3\eta_1(\chi,\tau)\right| \left|\vartheta'(\tau\chi + (1-\tau)\kappa_2)\right| d\tau \\ &\quad + \frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\int_0^1 \left|\nu_1(\chi,1) - 3\nu_1(\chi,\tau)\right| \left|\vartheta'((1-\tau)\kappa_1 + \tau\chi)\right| d\tau. \end{aligned} \tag{6}$$

With the help of the convexity of $\left|\vartheta'\right|$, we get

$$\begin{aligned} &\left|\frac{1}{6}\left[\vartheta(\kappa_1) + 4\vartheta(\chi) + \vartheta(\kappa_2)\right] - \frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)} + \frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\right| \\ &\le \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\int_0^1 \left|\eta_1(\chi,1) - 3\eta_1(\chi,\tau)\right| \left[\tau\left|\vartheta'(\chi)\right| + (1-\tau)\left|\vartheta'(\kappa_2)\right|\right] d\tau \\ &\quad + \frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\int_0^1 \left|\nu_1(\chi,1) - 3\nu_1(\chi,\tau)\right| \left[(1-\tau)\left|\vartheta'(\kappa_1)\right| + \tau\left|\vartheta'(\chi)\right|\right] d\tau \\ &= \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left[\Xi_1\left|\vartheta'(\chi)\right| + \Xi_2\left|\vartheta'(\kappa_2)\right|\right] + \frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left[\Xi_3\left|\vartheta'(\kappa_1)\right| + \Xi_4\left|\vartheta'(\chi)\right|\right]. \end{aligned}$$

This completes the proof of Theorem 2. □

Corollary 5. *Under assumptions of Theorem 2 with* $\chi = \frac{\kappa_1+\kappa_2}{2}$*, we have the following inequalities*

$$\begin{aligned} &\left|\frac{1}{6}\left[\vartheta(\kappa_1) + 4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right) + \vartheta(\kappa_2)\right]\right. \\ &\quad \left. - \frac{1}{2\Upsilon_1(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\vartheta(\kappa_2) + {}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\vartheta(\kappa_1)\right]\right| \end{aligned}$$

$$\leq \frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left[2\Xi_5\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|+\Xi_6\left[\left|\vartheta'(\kappa_2)\right|+\left|\vartheta'(\kappa_1)\right|\right]\right]$$
$$\leq \frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}(\Xi_5+\Xi_6)\left[\left|\vartheta'(\kappa_2)\right|+\left|\vartheta'(\kappa_1)\right|\right].$$

Here,

$$\Xi_5=\int_0^1 \tau\left|\Upsilon_1(1)-3\Upsilon_1(\tau)\right|d\tau \text{ and } \Xi_6=\int_0^1 (1-\tau)\left|\Upsilon_1(1)-3\Upsilon_1(\tau)\right|d\tau. \tag{7}$$

Corollary 6. *In Theorem 2, let us note that $\varphi(\tau)=\tau$ for all $\tau\in[\kappa_1,\kappa_2]$. Then, we obtain the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{1}{\kappa_2-\chi}\int_\chi^{\kappa_2}\vartheta(\tau)d\tau+\frac{1}{\chi-\kappa_1}\int_{\kappa_1}^{\chi}\vartheta(\tau)d\tau\right]\right|$$
$$\leq \frac{\kappa_2-\chi}{6}\left[\frac{29}{54}\left|\vartheta'(\chi)\right|+\frac{8}{27}\left|\vartheta'(\kappa_2)\right|\right]+\frac{\chi-\kappa_1}{6}\left[\frac{8}{27}\left|\vartheta'(\kappa_1)\right|+\frac{29}{54}\left|\vartheta'(\chi)\right|\right].$$

Corollary 7. *In Theorem 2, if we select $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)}, \alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$, then we get the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{J^\alpha_{\chi+}\vartheta(\kappa_2)}{(\kappa_2-\chi)^\alpha}+\frac{J^\alpha_{\chi-}\vartheta(\kappa_1)}{(\chi-\kappa_1)^\alpha}\right]\right|$$
$$\leq \frac{\kappa_2-\chi}{6}\left[\Theta_1(\alpha)\left|\vartheta'(\chi)\right|+\Theta_2(\alpha)\left|\vartheta'(\kappa_2)\right|\right]$$
$$+\frac{\chi-\kappa_1}{6}\left[\Theta_2(\alpha)\left|\vartheta'(\kappa_1)\right|+\Theta_1(\alpha)\left|\vartheta'(\chi)\right|\right],$$

where

$$\Theta_1(\alpha)=\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{4-\alpha}{2(\alpha+2)}, \tag{8}$$
$$\Theta_2(\alpha)=\frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}+\frac{4-3\alpha-\alpha^2}{2(\alpha+1)(\alpha+2)}.$$

Corollary 8. *In Theorem 2, consider $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, k,\alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$, then we have the following inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma_k(\alpha+k)}{2}\left[\frac{J^\alpha_{\chi+,k}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\frac{\alpha}{k}}}+\frac{J^\alpha_{\chi-,k}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\frac{\alpha}{k}}}\right]\right|$$
$$\leq \frac{\kappa_2-\chi}{6}\left[\Psi_1(\alpha,k)\left|\vartheta'(\chi)\right|+\Psi_2(\alpha,k)\left|\vartheta'(\kappa_2)\right|\right]$$
$$+\frac{\chi-\kappa_1}{6}\left[\Psi_2(\alpha,k)(\alpha)\left|\vartheta'(\kappa_1)\right|+\Psi_1(\alpha,k)\left|\vartheta'(\chi)\right|\right].$$

Here,

$$\Psi_1(\alpha,k)=\frac{\alpha}{\alpha+2k}\left(\frac{1}{3}\right)^{\frac{2k}{\alpha}}+\frac{4k-\alpha}{2(\alpha+2k)},\tag{9}$$

$$\Psi_2(\alpha,k)=\frac{2\alpha}{\alpha+k}\left(\frac{1}{3}\right)^{\frac{k}{\alpha}}-\frac{\alpha}{\alpha+2k}\left(\frac{1}{3}\right)^{\frac{2k}{\alpha}}+\frac{4k^2-3\alpha k-\alpha^2}{2(\alpha+k)(\alpha+2k)}.$$

Remark 2. *If we set $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 6, then Corollary 6 reduces to [43, Corollary 1].*

Remark 3. *Assume $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 7. Then, we obtain the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.\\&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\vartheta(\kappa_1)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{12}\left[\Theta_2(\alpha)\left(\left|\vartheta'(\kappa_2)\right|+\left|\vartheta'(\kappa_1)\right|\right)+2\Theta_1(\alpha)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\right],\end{aligned}$$

which is given by Hai and Wang in [23].

Remark 4. *Assume $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 8. Then, we obtain the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.\\&\quad\left.-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\vartheta(\kappa_1)\right]\right|\\&\leq\frac{\kappa_2-\kappa_1}{12}\left[\Psi_2(\alpha,k)\left(\left|\vartheta'(\kappa_2)\right|+\left|\vartheta'(\kappa_1)\right|\right)+2\Psi_1(\alpha,k)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|\right].\end{aligned}$$

Theorem 3. *Suppose that the assumptions of Lemma 1 hold. Suppose also that the mapping $|\vartheta'|^q$, $q>1$, is convex on $[\kappa_1,\kappa_2]$. Then, we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta(\kappa_2)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta(\kappa_1)}{\nu_1(\chi,1)}\right]\right|\\&\leq\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\\&\quad+\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Hölder inequality (6), we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta\left(\kappa_{1}\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_{2}\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_{2}\right)}{\eta_{1}(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_{1}\right)}{\nu_{1}(\chi,1)}\right]\right| \\
&\leq \frac{\kappa_{2}-\chi}{6\eta_{1}(\chi,1)}\left(\int_{0}^{1}\left|\eta_{1}(\chi,1)-3\eta_{1}(\chi,\tau)\right|^{p}d\tau\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\vartheta'\left(\tau\chi+\left(1-\tau\right)\kappa_{2}\right)\right|^{q}d\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{\chi-\kappa_{1}}{6\nu_{1}\left(\chi,1\right)}\left(\int_{0}^{1}\left|\nu_{1}(\chi,1)-3\nu_{1}(\chi,\tau)\right|^{p}d\tau\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\vartheta'\left(\left(1-\tau\right)\kappa_{1}+\tau\chi\right)\right|^{q}d\tau\right)^{\frac{1}{q}}.
\end{aligned}$$

By using convexity of $\left|\vartheta'\right|^{q}$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta\left(\kappa_{1}\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_{2}\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_{2}\right)}{\eta_{1}(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_{1}\right)}{\nu_{1}(\chi,1)}\right]\right| \\
&\leq \frac{\kappa_{2}-\chi}{6\eta_{1}(\chi,1)}\left(\int_{0}^{1}\left|\eta_{1}(\chi,1)-3\eta_{1}(\chi,\tau)\right|^{p}d\tau\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_{0}^{1}\left(\tau\left|\vartheta'(\chi)\right|^{q}+\left(1-\tau\right)\left|\vartheta'(\kappa_{2})\right|^{q}\right)d\tau\right)^{\frac{1}{q}} \\
&\quad+\frac{\chi-\kappa_{1}}{6\nu_{1}\left(\chi,1\right)}\left(\int_{0}^{1}\left|\nu_{1}(\chi,1)-3\nu_{1}(\chi,\tau)\right|^{p}d\tau\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_{0}^{1}\left(\left(1-\tau\right)\left|\vartheta'\left(\kappa_{1}\right)\right|^{q}+\tau\left|\vartheta'\left(\chi\right)\right|^{q}\right)d\tau\right)^{\frac{1}{q}} \\
&=\frac{\kappa_{2}-\chi}{6\eta_{1}(\chi,1)}\left(\int_{0}^{1}\left|\eta_{1}(\chi,1)-3\eta_{1}(\chi,\tau)\right|^{p}d\tau\right)^{\frac{1}{p}}\left(\frac{\left|\vartheta'(\chi)\right|^{q}+\left|\vartheta'(\kappa_{2})\right|^{q}}{2}\right)^{\frac{1}{q}} \\
&\quad+\frac{\chi-\kappa_{1}}{6\nu_{1}\left(\chi,1\right)}\left(\int_{0}^{1}\left|\nu_{1}(\chi,1)-3\nu_{1}(\chi,\tau)\right|^{p}d\tau\right)^{\frac{1}{p}}\left(\frac{\left|\vartheta'(\chi)\right|^{q}+\left|\vartheta'(\kappa_{1})\right|^{q}}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof of Theorem 3. □

Corollary 9. *Under assumptions of Theorem 3 with* $\chi = \frac{\kappa_1+\kappa_2}{2}$, *we have the following inequalities*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.\\
&\quad\left.-\frac{1}{2\Upsilon_1(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\vartheta(\kappa_2)+{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\vartheta(\kappa_1)\right]\right|\\
&\leq \frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\int_0^1 |\Upsilon_1(1)-3\Upsilon_1(\tau)|^p\,d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\right]\\
&\leq \frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\int_0^1 |\Upsilon_1(1)-3\Upsilon_1(\tau)|^p\,d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{\left|\vartheta'(\kappa_1)\right|^q+3\left|\vartheta'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|\vartheta'(\kappa_1)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 10. *In Theorem 3, let us consider* $\varphi(\tau)=\tau$ *for all* $\tau\in[\kappa_1,\kappa_2]$. *Then, we obtain the inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{1}{\kappa_2-\chi}\int_{\chi}^{\kappa_2}\vartheta(\tau)d\tau+\frac{1}{\chi-\kappa_1}\int_{\kappa_1}^{\chi}\vartheta(\tau)d\tau\right]\right|\\
&\leq \frac{1}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\\
&\quad\times\left[(\kappa_2-\chi)\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}+(\chi-\kappa_1)\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 11. *In Theorem 3, if we take* $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}, \alpha>0$ *for all* $\tau\in[\kappa_1,\kappa_2]$, *then we get the inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{J_{\chi+}^{\alpha}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\alpha}}+\frac{J_{\chi-}^{\alpha}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\alpha}}\right]\right|\\
&\leq \frac{1}{6}\left(\int_0^1 |1-3\tau^{\alpha}|^p\,d\tau\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left[(\kappa_2-\chi)\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}+(\chi-\kappa_1)\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Corollary 12. *In Theorem 3, let us note that* $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, k,\alpha>0$ *for all* $\tau\in[\kappa_1,\kappa_2]$. *Then, we have the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma_k(\alpha+k)}{2}\left[\frac{J^{\alpha}_{\chi+,k}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\frac{\alpha}{k}}}+\frac{J^{\alpha}_{\chi-,k}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\frac{\alpha}{k}}}\right]\right|$$

$$\leq\frac{1}{6}\left(\int_0^1\left|1-3\tau^{\frac{\alpha}{k}}\right|^p d\tau\right)^{\frac{1}{p}}$$

$$\times\left[(\kappa_2-\chi)\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}+(\chi-\kappa_1)\left(\frac{\left|\vartheta'(\chi)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Remark 5. *If we assign* $\chi=\frac{\kappa_1+\kappa_2}{2}$ *in Corollary 10, then Corollary 10 reduces to [43, Corollary 3].*

Remark 6. *Consider* $\chi=\frac{\kappa_1+\kappa_2}{2}$ *in Corollaries 11 and 12. Then, we obtain the following inequalities*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\vartheta(\kappa_1)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{12}\left(\int_0^1\left|1-3\tau^{\alpha}\right|^p d\tau\right)^{\frac{1}{p}}\left[\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\right]$$

and

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\vartheta(\kappa_1)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{12}\left(\int_0^1\left|1-3\tau^{\frac{\alpha}{k}}\right|^p d\tau\right)^{\frac{1}{p}}\left[\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\left|\vartheta'(\kappa_1)\right|^q}{2}\right)^{\frac{1}{q}}\Bigg],$$

respectively.

Theorem 4. *Suppose that the assumptions of Lemma 1 hold. If the mapping $\left|\vartheta'\right|^q$, $q\geq 1$, is convex on $[\kappa_1,\kappa_2]$, then we have the following inequality*

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_2\right)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_1\right)}{\nu_1(\chi,1)}\right]\right|$$
$$\leq\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|d\tau\right)^{1-\frac{1}{q}}\left(\Xi_1\left|\vartheta'\left(\chi\right)\right|^q+\Xi_2\left|\vartheta'\left(\kappa_2\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6\nu_1\left(\chi,1\right)}\left(\int_0^1\left|\nu_1\left(\chi,1\right)-3\nu_1\left(\chi,\tau\right)\right|d\tau\right)^{1-\frac{1}{q}}\left(\Xi_3\left|\vartheta'\left(\kappa_1\right)\right|^q+\Xi_4\left|\vartheta'\left(\chi\right)\right|^q\right)^{\frac{1}{q}},$$

where Ξ_i, $i=1,2,3,4$ are defined as in equality (5).

Proof. By applying power mean inequality (6), we get

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_2\right)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_1\right)}{\nu_1(\chi,1)}\right]\right|$$
$$\leq\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|\left|\vartheta'\left(\tau\chi+(1-\tau)\kappa_2\right)\right|^q d\tau\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6\nu_1\left(\chi,1\right)}\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|\left|\vartheta'\left((1-\tau)\kappa_1+\tau\chi\right)\right|^q d\tau\right)^{\frac{1}{q}}.$$

Since $\left|\vartheta'\right|^q$ is convex, we obtain

$$\left|\frac{1}{6}\left[\vartheta\left(\kappa_1\right)+4\vartheta\left(\chi\right)+\vartheta\left(\kappa_2\right)\right]-\frac{1}{2}\left[\frac{{}_{\chi+}I_{\varphi}\vartheta\left(\kappa_2\right)}{\eta_1(\chi,1)}+\frac{{}_{\chi-}I_{\varphi}\vartheta\left(\kappa_1\right)}{\nu_1(\chi,1)}\right]\right|$$

$$\leq \frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1 \left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right| d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left[\tau\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|\left|\vartheta'(\chi)\right|^q\right.\right.$$
$$\left.\left.+(1-\tau)\left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right|\left|\vartheta'(\kappa_2)\right|^q\right] d\tau\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left(\int_0^1 \left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right| d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 \left[(1-\tau)\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|\left|\vartheta'(\kappa_1)\right|^q\right.\right.$$
$$\left.\left.+\tau\left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right|\left|\vartheta'(\chi)\right|^q d\tau\right]\right)^{\frac{1}{q}}$$
$$=\frac{\kappa_2-\chi}{6\eta_1(\chi,1)}\left(\int_0^1 \left|\eta_1(\chi,1)-3\eta_1(\chi,\tau)\right| d\tau\right)^{1-\frac{1}{q}}\left(\Xi_1\left|\vartheta'(\chi)\right|^q+\Xi_2\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6\nu_1(\chi,1)}\left(\int_0^1 \left|\nu_1(\chi,1)-3\nu_1(\chi,\tau)\right| d\tau\right)^{1-\frac{1}{q}}\left(\Xi_3\left|\vartheta'(\kappa_1)\right|^q+\Xi_4\left|\vartheta'(\chi)\right|^q\right)^{\frac{1}{q}}.$$

This completes the proof of Theorem 4. □

Corollary 13. *Under assumptions of Theorem 4 with* $\chi=\frac{\kappa_1+\kappa_2}{2}$, *we have the following inequalities*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{1}{2\Upsilon_1(1)}\left[{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\vartheta(\kappa_2)+{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\vartheta(\kappa_1)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\int_0^1 \left|\Upsilon_1(1)-3\Upsilon_1(\tau)\right| d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\Xi_5\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\Xi_6\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Xi_6\left|\vartheta'(\kappa_1)\right|^q+\Xi_5\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\kappa_2-\kappa_1}{12\Upsilon_1(1)}\left(\int_0^1 |\Upsilon_1(1)-3\Upsilon_1(\tau)|\,d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{\Xi_5\left|\vartheta'(\kappa_1)\right|^q+(\Xi_5+2\Xi_6)\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\Xi_5+2\Xi_6)\left|\vartheta'(\kappa_1)\right|^q+\Xi_5\left|\vartheta'(\kappa_2)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Here, Ξ_5 and Ξ_6 are defined as in equality (7).

Corollary 14. *In Theorem 4, if we choose $\varphi(\tau)=\tau$ for all $\tau\in[\kappa_1,\kappa_2]$, then we obtain the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{1}{2}\left[\frac{1}{\kappa_2-\chi}\int_\chi^{\kappa_2}\vartheta(\tau)d\tau+\frac{1}{\chi-\kappa_1}\int_{\kappa_1}^{\chi}\vartheta(\tau)d\tau\right]\right|$$
$$\leq\frac{\kappa_2-\chi}{6}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\frac{29}{54}\left|\vartheta'(\chi)\right|^q+\frac{8}{27}\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6}\left(\frac{5}{6}\right)^{1-\frac{1}{q}}\left(\frac{8}{27}\left|\vartheta'(\kappa_1)\right|^q+\frac{29}{54}\left|\vartheta'(\chi)\right|^q\right)^{\frac{1}{q}}.$$

Corollary 15. *In Theorem 4, let us note that $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)},\alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$. Then, we have the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{J_{\chi+}^{\alpha}\vartheta(\kappa_2)}{(\kappa_2-\chi)^\alpha}+\frac{J_{\chi-}^{\alpha}\vartheta(\kappa_1)}{(\chi-\kappa_1)^\alpha}\right]\right|$$
$$\leq\frac{\kappa_2-\chi}{6}\left(\Theta_3(\alpha)\right)^{1-\frac{1}{q}}\left(\Theta_1(\alpha)\left|\vartheta'(\chi)\right|^q+\Theta_2(\alpha)\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6}\left(\Theta_3(\alpha)\right)^{1-\frac{1}{q}}\left(\Theta_2(\alpha)\left|\vartheta'(\kappa_1)\right|^q+\Theta_1(\alpha)\left|\vartheta'(\chi)\right|^q\right)^{\frac{1}{q}},$$

where $\Theta_i(\alpha)$, $i=1,2$ are defined as in equality (8) and

$$\Theta_3(\alpha)=2\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\left[1-\frac{1}{\alpha+1}\right]+\frac{3}{\alpha+1}-1.$$

Corollary 16. *In Theorem 4, if we set $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},k,\alpha>0$ for all $\tau\in[\kappa_1,\kappa_2]$, then we get the inequality*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta(\chi)+\vartheta(\kappa_2)\right]-\frac{\Gamma_k(\alpha+k)}{2}\left[\frac{J_{\chi+,k}^{\alpha}\vartheta(\kappa_2)}{(\kappa_2-\chi)^{\frac{\alpha}{k}}}+\frac{J_{\chi-,k}^{\alpha}\vartheta(\kappa_1)}{(\chi-\kappa_1)^{\frac{\alpha}{k}}}\right]\right|$$

$$\leq \frac{\kappa_2-\chi}{6}\left(\Psi_3(\alpha,k)\right)^{1-\frac{1}{q}}\left(\Psi_1(\alpha,k)\left|\vartheta'(\chi)\right|^q+\Psi_2(\alpha,k)\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\kappa_1}{6}\left(\Psi_3(\alpha,k)\right)^{1-\frac{1}{q}}\left(\Psi_2(\alpha,k)\left|\vartheta'(\kappa_1)\right|^q+\Psi_1(\alpha,k)\left|\vartheta'(\chi)\right|^q\right)^{\frac{1}{q}},$$

where $\Psi_i(\alpha,k)$, $i=1,2$ are defined as in equality (9) and

$$\Psi_3(\alpha,k)=2\left(\frac{1}{3}\right)^{\frac{k}{\alpha}}\left[1-\frac{k}{\alpha+k}\right]+\frac{3k}{(\alpha+k)}-1.$$

Remark 7. *Considering $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollary 14, then Corollary 14 reduces to [43, Theorem 10 (for $s=1$)].*

Remark 8. *If we take $\chi=\frac{\kappa_1+\kappa_2}{2}$ in Corollaries 15 and 16, then we obtain the following inequalities*

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\vartheta(\kappa_1)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{12}\left(\Theta_3(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\Theta_1(\alpha)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\Theta_2(\alpha)\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Theta_2(\alpha)\left|\vartheta'(\kappa_1)\right|^q+\Theta_1(\alpha)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

and

$$\left|\frac{1}{6}\left[\vartheta(\kappa_1)+4\vartheta\left(\frac{\kappa_1+\kappa_2}{2}\right)+\vartheta(\kappa_2)\right]\right.$$
$$\left.-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\vartheta(\kappa_2)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\vartheta(\kappa_1)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{12}\left(\Psi_3(\alpha,k)\right)^{1-\frac{1}{q}}\left[\left(\Psi_1(\alpha,k)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q+\Psi_2(\alpha,k)\left|\vartheta'(\kappa_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Psi_2(\alpha,k)\left|\vartheta'(\kappa_1)\right|^q+\Psi_1(\alpha,k)\left|\vartheta'\left(\frac{\kappa_1+\kappa_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right],$$

respectively.

4. Conclusion

In this paper, we used the concepts of fractional calculus and proved some new inequalities of Simpson's type inequalities for differentiable convex mappings. Moreover, we discussed the special cases of the main results and several new inequalities of Simpson's type for differentiable convex functions via the ordinary integral are obtained. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for co-ordinated convex functions in their future research.

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