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SOME NOTES ON LIFTS OF THE $F\left((\upsilon+1),\lambda^2(\upsilon-1)\right)$-STRUCTURE ON COTANGENT AND TANGENT BUNDLE

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Abstract. The $F^{\upsilon+1}-\lambda^2F^{\upsilon-1}=0$ structure $(\upsilon\geq 3)$ have been studied by Kim J. B. [14]. Later, Srivastava S.K studied on the complete lifts of $(1,1)$ tensor field F satisfying structure $F^{\upsilon+1}-\lambda^2F^{\upsilon-1}=0$ and extended in M^n to cotangent bundle. This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the complete and horizontal lifts of $F^{\upsilon+1}-\lambda^2F^{\upsilon-1}=0$. Later, we get the results of Tachibana operators applied to vector and covector fields according to the complete and horizontal lifts of $F\left((\upsilon+1),\lambda^2(\upsilon-1)\right)$ -structure and the conditions of almost holomorfic vector fields in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of $F^{\upsilon+1}-\lambda^2F^{\upsilon-1}=0-$structure. In the second part, all results obtained in the first section were investigated according to the complete and horizontal lifts of the $F^{\upsilon+1}-\lambda^2F^{\upsilon-1}=0$ structure in tangent bundle $T(M^n)$.

1. Introduction

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [25]. Later, a lot of authors studied on the topics of the bundle, Riemannian manifolds and F structure too [1, 2, 3, 4, 5, 10, 12, 13, 16, 23]. There are a lot of structures in tangent and cotangent bundle. One of them is the $F\left((\upsilon+1),\lambda^2(\upsilon-1)\right)$-structure $(\upsilon\geq 3)$ have been studied by Kim J. B. [14].

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Later, Srivastava S.K studied on the complete lifts of $(1,1)$ tensor field F satisfying structure $F^{v+1}-\lambda^2F^{v-1}=0$ and extended in M^n to cotangent bundle [21]. In this context, a differentiable structure $F^{2v+4}+F^2=0$, $(F\neq 0, v\neq 0)$ studied by K.K. Dube [11] and Upadhyay and Gupta have obtained some integrability conditions of $F(K,-(K-2))-$structure, satisfying $F^K+F^{K-2}=0$,(F is a tensor field of type $(1,1)$) [24].

This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the complete and horizontal lifts of $F\left((v+1),\lambda^2(v-1)\right)$-structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the complete and horizontal lifts of $F^{v+1}-\lambda^2F^{v-1}=0$ structure and the conditions of almost holomorfic vector fields in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of $F^{v+1}-\lambda^2F^{v-1}=0$ structure. In the second part, all results obtained in the first section were investigated according to the complete and horizontal lifts of the $F\left((v+1),\lambda^2(v-1)\right)$-structure in tangent bundle $T(M^n)$.

Let M^n be a differentiable manifold of class C^∞ and of dimension n and let $T^*(M^n)$ denote the cotangent bundle of M. Then $T^*(M^n)$ is also a differentiable manifold of class C^∞ and dimension $2n$.

The following are notations and conventions that will be used in this paper.

(1) $\Im_s^r(M^n)$ denotes the set of the tensor fields C^∞ and of type (r,s) on M^n. Similarly, $\Im_s^r(T^*(M^n))$ denotes the set of such tensor fields in $T^*(M^n)$.
(2) The map π is the projection of $T^*(M^n)$ onto M^n.
(3) Vector fields in M^n are denoted by X, Y, Z,...and Lie differentiation by L_X.The Lie product of vector fields X and Y is denoted by $[X,Y]$.
(4) Suffixes a,b,c,...,h,i,j... take the values 1 to n and $\bar{\imath}=i+n$. Suffixes A,B,C,...take the values 1 to $2n$.

If A is point in M^n, then $(\pi^*)^{-1}(A):M^n\longrightarrow T^*(M^n)$ is fiber over A. Any point $p\in(\pi^*)^{-1}(A)$ is denoted by the ordered pair (A,p_A), where p is $1-$form in M^n and p_A is the value of p at A. Let U be a coordinate neighborhood in M^n such that $A\in U$. Then U induces a coordinate neighborhood $(\pi^*)^{-1}(U)$ in $T^*(M^n)$ and $p\in\pi^{-1}(A)$.

1.1. **The complete lift of $F^{v+1}-\lambda^2F^{v-1}=0$ on cotangent bundle.** Let M^n be an n-dimensional connected differentiable manifold of class C^∞. Let there be given in M^n, a $(1,1)$ tensor field F of class C^∞ satisfying [14,21]

$$F^{v+1}-\lambda^2F^{v-1}=0, \tag{1}$$

where λ is non zero complex number. Also

$$\begin{aligned} rank\,(F) &= \frac{1}{2}\left(rank\ F^{v+1}+\dim\ M^n\right) &(2)\\ &= (a\ cons\tan t\ every\ where\ on\ M^n) &(3)\end{aligned}$$

Let the operators l^* and m^* be defined as

$$l^* def\,(F/\lambda)^{\upsilon-1}, m^* = I - (F/\lambda)^{\upsilon-1}, \tag{4}$$

where I denotes the identity operator on M^n. Then the operators I^* and m^* applied to the tangent space at a point of the manifold be complementary projection operators.

Let F_i^h be the component of F at A in the coordinate neighbourhood U of M^n. Then the complete lift F^C of F is also a tensor field of type $(1,1)$ in $T^*(M^n)$ whose components $\tilde{F}_B^A$ in $(\pi^*)^{-1}(U) : M^n \longrightarrow T^*(M^n)$ are given by [17]

$$\tilde{F}_i^h = F_i^h, \tag{5}$$

$$\tilde{F}_{\bar{i}}^h = 0, \tag{6}$$

$$\tilde{F}_i^{\bar{h}} = p_a[\partial F_h^a/\partial x^i - \partial F_i^a/\partial x^h] \tag{7}$$

and

$$\tilde{F}_{\bar{i}}^{\bar{h}} = F_h^i, \tag{8}$$

where $(x^1, x^2, x^3, ..., x^n)$ are coordinates of A in U and p_A has components $(p_1, p_2, p_3, ...p_n)$. Thus we can write

$$F^C = (\tilde{F}_B^A) = \begin{bmatrix} F_i^h & 0 \\ p_a(\partial_i F_h^a - \partial_h F_i^a) & F_h^i \end{bmatrix}, \tag{9}$$

where $\partial_i = \partial/\partial x^i$.

If we put

$$\partial_i F_h^a - \partial_h F_i^a = 2\partial[iF_h^a], \tag{10}$$

then the equation (9) can be written as

$$F^C = (\tilde{F}_B^A) = \begin{bmatrix} F_i^h & 0 \\ 2p_a\partial[iF_h^a] & F_h^i \end{bmatrix} \tag{11}$$

$$(F^C)^2 = (F^C)(F^C) = \begin{bmatrix} F_i^h F_j^i & 0 \\ L_{hj} & F_i^j F_h^i \end{bmatrix}. \tag{12}$$

Squaring (12) again we get [17]

$$\begin{aligned} (F^C)^4 &= \begin{bmatrix} F_i^h F_j^i & 0 \\ L_{hj} & F_i^j F_h^i \end{bmatrix} \begin{bmatrix} F_i^h F_j^i & 0 \\ L_{hj} & F_i^j F_h^i \end{bmatrix}, \\ &= \begin{bmatrix} F_i^h F_j^i F_l^k & 0 \\ F_k^j F_l^k L_{hj} + F_i^j F_h^i L_{jl} & F_k^l F_j^k F_i^j F_h^i \end{bmatrix}. \end{aligned}$$

$$(F^C)^6 = (F^C)^4 (F^C)^2 = \begin{pmatrix} F_i^h F_j^i F_k^j F_l^k F_m^l F_n^m & 0 \\ Q_{hn} & F_m^n F_l^m F_k^l F_j^k F_i^j F_h^i \end{pmatrix} \tag{13}$$

$$(F^C)^7 = (F^C)^6 (F^C) = \begin{pmatrix} -\lambda^2 F_p^n & 0 \\ -\lambda^2 p_s \partial[pF_h^s] & -\lambda^2 F_h^p \end{pmatrix}. \tag{14}$$

Thus it follows that

$$(F^{\upsilon+1})^C - \lambda^2 (F^{\upsilon-1})^C = 0 \tag{15}$$

Thus, we get the complete lift of $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0-$structure on the cotangent bundle.

1.2. **Horizontal lift of the structure $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0$ on cotangent bundle.** Let F, G be two tensor field of type $(1,1)$ on the manifold M^n. If F^H denotes the horizontal lift of F, we have [17, 18, 25]

$$F^H G^H + G^H F^H = (FG + GF)^H. \tag{16}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H. \tag{17}$$

Thus, multiplying both sides by F^H and making use of the same (16), we get

$$(F^H)^3 = (F^3)^H. \tag{18}$$

Thus it follows that

$$(F^H)^4 = (F^4)^H, \quad (F^H)^5 = (F^5)^H. \tag{19}$$

Thus,

$$(F^{\upsilon+1})^H - \lambda^2 (F^{\upsilon-1})^H = 0. \tag{20}$$

In view of (17), we can write $(F^H)^{\upsilon+1} - \lambda^2 (F^H)^{\upsilon-1} = 0$.

Thus, we get the horizontal lift of $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0-$structure on the cotangent bundle.

Proposition 1. *Let M^n be a Riemannian manifold with metric g, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle $T^*(M^n)$ of M^n satisfies the following*

$$\begin{aligned} i)\ [\omega^v, \theta^v] &= 0, \\ ii)\ \left[X^H, \omega^v\right] &= (\nabla_X \omega)^v, \\ iii)\ \left[X^H, Y^H\right] &= [X,Y]^H + \gamma R(X,Y) = [X,Y]^H + (pR(X,Y))^v \end{aligned} \tag{21}$$

for all $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$. (See [25] p. 238, p. 277 for more details).

2. Main Results

2.1. **The Nijenhuis tensors of $F\left((\upsilon+1), \lambda^2(\upsilon-1)\right)$-structure on cotangent bundle.**

Definition 2. *Let F be a tensor field of type $(1,1)$ satisfying $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0$ in M^n. The Nijenhuis tensor of a $(1,1)$ tensor field F of M^n is given by*

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y] \tag{22}$$

for any $X, Y \in \Im_0^1(M^n)$ [6, 19, 22]. The condition of $N_F(X,Y) = N(X,Y) = 0$ is essential to integrability condition in these structures.

The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^k \partial_k = (F_i^s \partial_s^k F_j^k - F_j^l \partial_l F_i^k - \partial_i F_j^l F_l^k + \partial_j F_i^s F_s^k)\partial_k,$$

where $X = \partial_i$, $Y = \partial_j$, $F \in \Im_1^1(M^n)$.

Proposition 3. *If X, $Y \in \Im_0^1(M^n)$, ω, $\theta \in \Im_1^0(M^n)$ and $F, G \in \Im_1^1(M^n)$, then [25]*

$$\begin{aligned} [\omega^v, \theta^v] &= 0, \quad [\omega^v, \gamma F] = (\omega \circ F)^v, \quad [\gamma F, \gamma G] = \gamma[F, G], \\ [X^C, \omega^v] &= (L_X \omega)^v, \;\; [X^C, \gamma F] = \gamma(L_X F), \;\; [X^C, Y^C] = [X, Y]^C, \end{aligned} \tag{23}$$

where $\omega \circ F$ is a $1-$form defined by $(\omega \circ F)(Z) = \omega(FZ)$ for any $Z \in \Im_0^1(M^n)$ and L_X the Lie derivative in direction of X.

Theorem 4. *The Nijenhuis tensor $N_{(F^{\upsilon+1})^C (F^{\upsilon+1})^C}\left(X^C, \omega^v\right)$ of the complete lift of $F^{\upsilon+1}$ vanishes if the Lie derivative of the tensor field $F^{\upsilon-1}$ with respect to X is zero and F is an almost $\pi-$structure on M (see [19] p.46).*

Proof. In consequence of Definition 2 the Nijenhuis tensor of $F^{\upsilon+1}$ is given by

$$\begin{aligned} & N_{(F^{\upsilon+1})^C (F^{\upsilon+1})^C}\left(X^C, \omega^v\right) \\ = \; & [\left(F^{\upsilon+1}\right)^C X^C, \left(F^{\upsilon+1}\right)^C \omega^v] - \left(F^{\upsilon+1}\right)^C [\left(F^{\upsilon+1}\right)^C X^C, \omega^v] \\ & - \left(F^{\upsilon+1}\right)^C [X^C, \left(F^{\upsilon+1}\right)^C \omega^V] + \left(F^{\upsilon+1}\right)^C \left(F^{\upsilon+1}\right)^C \left[X^C, \omega^v\right] \\ = \; & \lambda^4 \{ \left[\left(F^{\upsilon-1} X\right)^C + \gamma L_X F^{\upsilon-1}, \left(\omega \circ F^{\upsilon-1}\right)^v \right] \\ & - \left(F^{\upsilon-1}\right)^C [\left(F^{\upsilon-1} X\right)^C + \gamma L_X F^{\upsilon-1}, \omega^v] \\ & - \left(F^{\upsilon-1}\right)^C [X^C, \left(\omega \circ F^{\upsilon-1}\right)^v] + \left(F^{2\upsilon-2}\right)^C (L_X \omega)^v \} \\ = \; & \lambda^4 \{ (L_{(F^{\upsilon-1} X)} \left(\omega \circ F^{\upsilon-1}\right))^v - \left(\left(\omega \circ F^{\upsilon-1}\right) \circ \left(L_X F^{\upsilon-1}\right)\right)^v \\ & - \left(\left(L_{F^{\upsilon-1} X} \omega\right) \circ \left(F^{\upsilon-1}\right)\right)^v - \left(\left(\omega \circ \left(L_X F^{\upsilon-1}\right)\right)^v \circ F^{\upsilon-1}\right)^v \\ & - \left(\left(L_X \left(\omega \circ F^{\upsilon-1}\right)\right) \circ \left(F^{\upsilon-1}\right)\right)^v + \left((L_X \omega) \circ \left(F^{2\upsilon-2}\right)\right)^v \} \end{aligned}$$

If the lie derivatives of the tensor field $F^{\upsilon-1}$ with respect to X is zero, then the equation takes the form

$$= \lambda^4 \{ \omega \circ \left(L_{F^{\upsilon-1} X} F^{\upsilon-1}\right)^v - \left(\left(\omega \circ L_X F^{\upsilon-1}\right) F^{\upsilon-1}\right)^v \}$$

Let F be almost $\pi-$structure on M then $F^2 = \lambda^2 I$, where I is unit tensor field. So $F^{\upsilon-1} = \lambda^2 I$ and we get

$$N_{(F^{\upsilon+1})^C (F^{\upsilon+1})^C}\left(X^C, \omega^v\right) = 0$$

The theorem is proved. □

Theorem 5. *The Nijenhuis tensor* $N_{(F^{\upsilon+1})^C(F^{\upsilon+1})^C}\left(\omega^V,\theta^V\right)$ *of the complete lift of* $F^{\upsilon+1}$ *vanishes.*

Proof. Because $[\omega^V,\theta^V]=0$ and $\omega\circ F^{\upsilon-1}\in\Im_1^0(M^n)$ on $T^*(M^n)$, the Nijenhuis tensor $N(\omega^V,\theta^V)$ for the complete lift of $F^{\upsilon+1}$ is vanishes. □

2.2. **Tachibana operators applied to vector and covector fields according to lifts of $F\left((\upsilon+1),\lambda^2(\upsilon-1)\right)$-structure on cotangent bundle.**

Definition 6. *Let* $\varphi\in\Im_1^1(M^n)$*, and* $\Im(M^n)=\sum_{r,s=0}^{\infty}\Im_s^r(M^n)$ *be a tensor algebra over R. A map* $\phi_\varphi\mid_{r+s\rangle 0}$: $\overset{*}{\Im}(M^n)\rightarrow\Im(M^n)$ *is called as Tachibana operator or* ϕ_φ*operator on* M^n *if*

a) ϕ_φ is $R-$linear,

b) $\phi_\varphi:\overset{*}{\Im}(M^n)\rightarrow\Im_{s+1}^r(M^n)$ for all r and s,

c) $\phi_\varphi(K\overset{C}{\otimes}L)=(\phi_\varphi K)\otimes L+K\otimes\phi_\varphi L$ for all $K,L\in\overset{*}{\Im}(M^n)$,

d) $\phi_{\varphi X}Y=-(L_Y\varphi)X$ for all $X,Y\in\Im_0^1(M^n)$, where L_Y is the Lie derivative in direction of Y (see [7,9,15]),

e)

$$\begin{aligned}(\phi_{\varphi X}\eta)Y &= (d(\imath_Y\eta))(\varphi X)-(d(\imath_Y(\eta o\varphi)))X+\eta((L_Y\varphi)X)\\ &= \phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)+\eta((L_Y\varphi)X)\end{aligned}$$

for all $\eta\in\Im_1^0(M^n)$ and $X,Y\in\Im_0^1(M^n)$, where $\imath_Y\eta=\eta(Y)=\eta\overset{C}{\otimes}Y,\overset{*}{\Im}{}_s^r(M^n)$ the module of all pure tensor fields of type (r,s) on M^n with respect to the affinor field, $\overset{C}{\otimes}$ is a tensor product with a contraction C [6,8,19](see [22] for applied to pure tensor field).

Remark 7. *If* $r=s=0$*, then from c*), *d*) *and e*) *of Definition 6 we have* $\phi_{\varphi X}(\imath_Y\eta)=\phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)$ *for* $\imath_Y\eta\in\Im_0^0(M^n)$*, which is not well-defined* $\phi_\varphi-$*operator. Different choices of* Y *and* η *leading to same function* $f=\imath_Y\eta$ *do get the same values. Consider* $M^n=R^2$ *with standard coordinates* x,y*. Let* $\varphi=\begin{pmatrix}0&1\\1&0\end{pmatrix}$*. Consider the function* $f=1$*. This may be written in many different ways as* $\imath_Y\eta$*. Indeed taking* $\eta=dx$*, we may choose* $Y=\frac{\partial}{\partial_x}$ *or* $Y=\frac{\partial}{\partial_x}+x\frac{\partial}{\partial_y}$*. Now the right-hand side of* $\phi_{\varphi X}(\imath_Y\eta)=\phi X(\imath_Y\eta)-X(\imath_{\varphi Y}\eta)$ *is* $(\phi X)1-0=0$ *in the first case, and* $(\phi X)1-Xx=-Xx$ *in the second case. For* $X=\frac{\partial}{\partial_x}$*, the latter expression is* $-1\neq 0$*. Therefore, we put* $r+s>0$ *[19].*

Remark 8. *From d*) *of Definition 6 we have*

$$\phi_{\varphi X}Y=[\varphi X,Y]-\varphi[X,Y]. \tag{24}$$

By virtue of

$$[fX,gY]=fg[X,Y]+f(Xg)Y-g(Yf)X \tag{25}$$

for any $f,g\in\Im_0^0(M^n)$, we see that $\phi_{\varphi X}Y$ is linear in X, but not Y [19].

Theorem 9. *Let $\left(F^{\upsilon+1}\right)^C$ be a tensor field of type $(1,1)$ on $T^*(M^n)$. If the Tachibana operator $\phi_{(F^{\upsilon+1})^C}$ applied to vector and covector fields according to the structure $\left(F^{\upsilon+1}\right)^C-\lambda^2\left(F^{\upsilon-1}\right)^C=0$ defined by (15) on $T^*(M^n)$, then we get the following results.*

$$
\begin{aligned}
i)\ \phi_{(F^{\upsilon+1})^C X^C}Y^C &= -\lambda^2\{\left(\left(L_Y F^{\upsilon-1}\right)X\right)^C+\gamma\left(L_Y\left(L_X F^{\upsilon-1}\right)\right)-\gamma\left(L_{[Y,X]}F^{\upsilon-1}\right)\},\\
ii)\ \phi_{(F^{\upsilon+1})^C X^C}\omega^v &= -\lambda^2\{-\left(L_{F^{\upsilon-1}X}\omega\right)^v+\left(\omega\circ\left(L_X F^{\upsilon-1}\right)\right)^v+\left((L_X\omega)\circ F^{\upsilon-1}\right)\},\\
iii)\ \phi_{(F^{\upsilon+1})^C\omega^V}X^C &= -\lambda^2\left(\omega\left(L_X F^{\upsilon-1}\right)\right)^v,\\
iv)\ \phi_{(F^{\upsilon+1})^C\omega^V}\theta^v &= 0,
\end{aligned}
$$

where the complete lifts $X^C,Y^C\in\Im_0^1\left(T^*(M^n)\right)$ of $X,Y\in\Im_0^1(M)$ and the vertical lift $\omega^v,\theta^v\in\Im_0^1\left(T^*(M^n)\right)$ of $\omega,\theta\in\Im_1^0(M)$ are given, respectively.

Proof. *i)*

$$
\begin{aligned}
\phi_{(F^{\upsilon+1})^C X^C}Y^C &= -(L_{Y^C}\left(F^{\upsilon+1}\right)^C)X^C\\
&= -L_{Y^C}\left(F^{\upsilon+1}\right)^C X^C+\left(F^{\upsilon+1}\right)^C L_{Y^C}X^C\\
&= -\lambda^2[Y^C,\left(F^{\upsilon-1}X\right)^C]-\lambda^2\left[Y^C,\gamma L_X F^{\upsilon-1}\right]\\
&\quad +\lambda^2(F^{\upsilon-1}[Y,X])^C+\lambda^2\gamma\left(L_{[Y,X]}\right)F^{\upsilon-1})\\
&= -\lambda^2\{\left(\left(L_Y F^{\upsilon-1}\right)X\right)^C+\gamma\left(L_Y\left(L_X F^{\upsilon-1}\right)\right)-\gamma\left(L_{[Y,X]}F^{\upsilon-1}\right)\}
\end{aligned}
$$

ii)

$$
\begin{aligned}
\phi_{(F^{\upsilon+1})^C X^C}\omega^v &= -(L_{\omega^V}\left(F^{\upsilon+1}\right)^C)X^C\\
&= -L_{\omega^V}\left(F^{\upsilon+1}\right)^C X^C+\left(F^{\upsilon+1}\right)^C L_{\omega^V}X^C\\
&= -\lambda^2([\omega^v,\left(F^{\upsilon-1}X\right)^C+\gamma\left(L_X F^{\upsilon-1}\right)])-\lambda^2\left(F^{\upsilon-1}\right)^C(L_X\omega)^v\\
&= -\lambda^2\{-\left(L_{F^{\upsilon-1}X}\omega\right)^v+\left(\omega\circ\left(L_X F^{\upsilon-1}\right)\right)^v+\left((L_X\omega)\circ F^{\upsilon-1}\right)\}
\end{aligned}
$$

iii)

$$
\begin{aligned}
\phi_{(F^{\upsilon+1})^C\omega^V}X^C &= -(L_{X^C}\left(F^{\upsilon+1}\right)^C)\omega^V\\
&= -L_{X^C}\left(F^{\upsilon+1}\right)^C\omega^V+\left(F^{\upsilon+1}\right)^C L_{X^C}\omega^V\\
&= -\lambda^2\left(L_X\left(\omega\circ F^{\upsilon-1}\right)\right)^V+\lambda^2\left((L_X\omega)\circ F^{\upsilon-1}\right)^V\\
&= -\lambda^2\left(\omega\left(L_X F^{\upsilon-1}\right)\right)^V
\end{aligned}
$$

iv)

$$
\begin{aligned}
\phi_{(F^{\upsilon+1})^C\omega^V}\theta^v &= -(L_{\theta^V}\left(F^{\upsilon+1}\right)^C)\omega^v\\
&= -L_{\theta^V}\left(F^{\upsilon+1}\right)^C\omega^V+\left(F^{\upsilon+1}\right)^C L_{\theta^V}\omega^v
\end{aligned}
$$

$$\begin{aligned} &= -\lambda^2 L_{\theta^V}\left(\omega\circ F^{\upsilon-1}\right)^v \\ &= 0 \end{aligned}$$

□

Proposition 10. *The complete lift Y^C is an holomorfic vector field with respect to the structure $\left(F^{\upsilon+1}\right)^C-\lambda^2\left(F^{\upsilon-1}\right)^C=0$, if $L_Y F^{\upsilon-1}=0$.*

Proof. i)

$$\begin{aligned} (L_{Y^C}\left(F^{\upsilon+1}\right)^C)X^C &= L_{Y^C}\left(F^{\upsilon+1}\right)^C X^C-\left(F^{\upsilon+1}\right)^C L_{Y^C}X^C \\ &= \lambda^2\left(\left(L_Y F^{\upsilon-1}\right)X\right)^C+\lambda^2\left(F^{\upsilon-1}\left(L_Y X\right)\right)^C \\ &\quad +\lambda^2\gamma\left(L_Y\left(L_X F^{\upsilon-1}\right)\right)-\lambda^2\left(F^{\upsilon-1}\left(L_Y X\right)\right)^C \\ &\quad -\lambda^2\gamma\left(L_{[Y,X]}F^{\upsilon-1}\right) \\ &= \lambda^2\{\left(\left(L_Y F^{\upsilon-1}\right)X\right)^C+\gamma\left(L_Y\left(L_X F^{\upsilon-1}\right)\right) \\ &\quad -\gamma\left(L_{[Y,X]}F^{\upsilon-1}\right)\} \end{aligned}$$

ii)

$$\begin{aligned} (L_{Y^C}\left(F^{\upsilon+1}\right)^C)\omega^v &= L_{Y^C}\left(F^{\upsilon+1}\right)^C\omega^v-\left(F^{\upsilon+1}\right)^C L_{Y^C}\omega^v \\ &= \lambda^2 L_Y\left(\omega\circ F^{\upsilon-1}\right)^v-\lambda^2\left(\left(L_Y\omega\right)\circ F^{\upsilon-1}\right)^v \\ &= \lambda^2\left(\omega\left(L_Y F^{\upsilon-1}\right)\right)^v \end{aligned}$$

where $Y\in\Im_0^1(M)$ and L_Y is the Lie derivative in direction of Y. □

2.3. **The purity conditions of Sasakian metric with respect to $(F^{\upsilon+1})^C$ on $T^*(M^n)$.** Let F be an affinor field on M^n, i.e. $F\in\Im_1^1(M^n)$. A tensor field t of (r,s) is called pure tensor field with respect to F if

$$\begin{aligned} t(FX_1,X_2,...,X_s,\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}) &= t(X_1,FX_2,...,X_s,\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}) \\ &\quad ... \\ &\quad ... \\ &\quad ... \\ &= t(X_1,X_2,...,FX_s,\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}) \\ &= t(X_1,X_2,...,X_s,{}'F\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}) \\ &= t(X_1,X_2,...,X_s,\overset{1}{\xi},{}'F\overset{2}{\xi},...,\overset{r}{\xi}) \\ &\quad ... \\ &\quad ... \\ &\quad ... \end{aligned}$$

$$= \quad t(X_1, X_2, ..., X_s, \overset{1}{\xi}, \overset{2}{\xi}, ..., {}'F\overset{r}{\xi})$$

for any $X_1, X_2, ..., X_s \in \Im_0^1(M^n)$ and $\overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi} \in \Im_1^0(M^n)$, where ${}'F$ is the adjoint operator of F defined by

$$({}'F\xi)(X) = \xi(FX) = (\xi o F)(X)$$

Definition 11. *A Sasakian metric ${}^S g$ is defined on $T^*(M^n)$ by the three equations [20]*

$${}^S g(\omega^v, \theta^v) = (g^{-1}(\omega, \theta))^v = g^{-1}(\omega, \theta) o\pi, \tag{26}$$

$${}^S g(\omega^v, Y^H) = 0, \tag{27}$$

$${}^S g(X^H, Y^H) = (g(X,Y))^v = g(X,Y) \circ \pi. \tag{28}$$

For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T_x^*(M^n)$ by

$$g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j, \tag{29}$$

where $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$. Since any tensor field of type $(0,2)$ on $T^*(M^n)$ is completely determined by its action on vector fields of type X^H and ω^v (see [25], p.280), it follows that ${}^S g$ is completely determined by equations (26), (27) and (28).

Theorem 12. *Let $(T^*(M^n), {}^S g)$ be the cotangent bundle equipped with Sasakian metric ${}^S g$ and a tensor field $(F^{v+1})^C$ of type $(1,1)$ defined by (15) on $T^*(M^n)$. Sasakian metric ${}^S g$ is pure with respect to $(F^{v+1})^C$ if $F^{v-1} = \lambda^2 I$ and $\nabla F^{v-1} = 0$. (I =identity tensor field of type $(1,1)$)*

Proof. We put

$$S(\tilde{X}, \tilde{Y}) = {}^S g((F^{v+1})^C \tilde{X}, \tilde{Y}) - {}^S g(\tilde{X}, (F^{v+1})^C \tilde{Y}).$$

If $S(\tilde{X}, \tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ω^v, θ^v or X^H, Y^H, then $S = 0$. By virtue of $(F^{v+1})^C - \lambda^2 (F^{v-1})^C = 0$ and (26),(27), (28), we get

i)

$$\begin{aligned}
S(\omega^v, \theta^v) &= {}^S g(\left(F^{v+1}\right)^C \omega^v, \theta^v) - {}^S g(\omega^v, \left(F^{v+1}\right)^C \theta^v), \\
&= {}^S g(\lambda^2 \left(F^{v-1}\right)^C \omega^v, \theta^v) - {}^S g(\omega^v, \lambda^2 \left(F^{v-1}\right)^C \theta^v), \\
&= \lambda^2 \{ {}^S g(\left(\omega \circ F^{v-1}\right)^v, \theta^v) - {}^S g(\omega^v, \left(\theta \circ F^{v-1}\right)^v)\}, \\
&= \lambda^2 \{ \left(g^{-1}\left(\left(\omega \circ F^{v-1}\right), \theta\right)\right)^v - \left(g^{-1}\left(\omega, \left(\theta \circ F^{v-1}\right)\right)\right)^v \}.
\end{aligned}$$

ii)

$$\begin{aligned}
S\left(X^H, \theta^v\right) &= {}^S g(\left(F^{v+1}\right)^C X^H, \theta^v) - {}^S g(X^H, \left(F^{v+1}\right)^C \theta^v), \\
&= {}^S g(\lambda^2 \left(F^{v-1}\right)^C X^H, \theta^v) - {}^S g(X^H, \left(F^{v-1}\right)^C \theta^v),
\end{aligned}$$

$$= \lambda^2 \ ({}^S g(\left(F^{\upsilon-1}X\right)^H, \theta^v)) + \lambda^2 \ ({}^S g((p\left[\nabla F^{\upsilon-1}\right]_X)^v, \theta^v)),$$
$$= \lambda^2 (g^{-1}((p\left[\nabla F^{\upsilon-1}\right]_X), \theta))^v,$$

where $\nabla_X F + F(\nabla_X) - \nabla F X = [\nabla F]_X$ (see [25] p. 279).
iii)

$$\begin{aligned} S\left(X^H, Y^H\right) &= {}^S g(\left(F^{\upsilon+1}\right)^C X^H, Y^H) - {}^S g(X^H, \left(F^{\upsilon+1}\right)^C Y^H) \\ &= \lambda^2 \ \{{}^S g(\left(F^{\upsilon-1}X\right)^H + \gamma(\left[\nabla F^{\upsilon-1}\right]_X), Y^H) \\ &\quad - {}^S g(X^H, \left(F^{\upsilon-1}Y\right)^H + \gamma(\left[\nabla F^{\upsilon-1}\right]_Y))\} \\ &= \lambda^2 \ \{ \ {}^S g(\left(F^{\upsilon-1}X\right)^H, Y^H) + {}^S g((p(\left[\nabla F^{\upsilon-1}\right]_X))^v, Y^H) \\ &\quad - {}^S g(X^H, \left(F^{\upsilon-1}Y\right)^H) - {}^S g(X^H, (p(\left[\nabla F^{\upsilon-1}\right]_Y))^v)\} \\ &= \lambda^2 \{\left(g\left(\left(F^{\upsilon-1}X\right), Y\right)\right)^v - \left(g\left(X, \left(F^{\upsilon-1}Y\right)\right)\right)^v\} \end{aligned}$$

where $F^C X^H = (FX)^H + \gamma([\nabla F]_X)$ for all $X^H \in \Im^1_0(T^*(M^n))$, $F^C \in \Im^1_1(T^*(M^n))$ and $[\nabla F]_X \in \Im^1_1(M^n)$ (see [25], p.279). □

2.4. **The structure** $(F^{\upsilon+1})^H - \lambda^r (F^{\upsilon-1})^H = 0$ **on cotangent bundle.** In this section, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of $F^{\upsilon+1} - \lambda^r F^{\upsilon-1} = 0$ structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of the structure $F^{\upsilon+1} - \lambda^r F^{\upsilon-1} = 0$ in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of $F^{\upsilon+1} - \lambda^r F^{\upsilon-1} = 0$ structure.

Theorem 13. *The Nijenhuis tensors of* $\left(F^{\upsilon+1}\right)^H$ *and* $F^{\upsilon-1}$ *denote by* $\tilde{N}$ *and* N*, respectively. Thus, taking account of the definition of the Nijenhuis tensor, the formulas (21) stated in Proposition 1 and the structure* $\left(F^{\upsilon+1}\right)^H - \lambda^2 \left(F^{\upsilon-1}\right)^H = 0$*, we find the following results of computation.*

$$\begin{aligned} i)\ \tilde{N}_{(F^{\upsilon+1})^H (F^{\upsilon+1})^H}\left(X^H, Y^H\right) &= \lambda^4 \{\left(N_{F^{\upsilon-1}F^{\upsilon-1}}\left(X, Y\right)\right)^H \\ &\quad + \gamma\{R\left(F^{\upsilon-1}X, F^{\upsilon-1}Y\right) - R\left(F^{\upsilon-1}X, Y\right) F^{\upsilon-1} \\ &\quad - R\left(X, F^{\upsilon-1}Y\right) F^{\upsilon-1} + R\left(X, Y\right)\left(F^{\upsilon-1}\right)^2\}\}, \end{aligned}$$

$$\begin{aligned} ii)\ \tilde{N}_{(F^{\upsilon+1})^H (F^{\upsilon+1})^H}\left(X^H, \omega^V\right) &= \lambda^4 \{\left(\omega\left(\nabla_{F^{\upsilon-1}X} F^{\upsilon-1}\right)\right)^v \\ &\quad - \left(\left(\omega\left(\nabla_X F^{\upsilon-1}\right)\right) F^{\upsilon-1}\right)^v\}, \end{aligned}$$

$$iii)\ \tilde{N}_{(F^{\upsilon+1})^H (F^{\upsilon+1})^H}\left(\omega^v, \theta^v\right) = 0.$$

Proof. The Nijenhuis tensor $N(X^H, Y^H)$ for the horizontal lift of $F^{\upsilon+1}$ is given by *i)*

$$\begin{aligned}
&\tilde{N}_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}\left(X^{H},Y^{H}\right)\\
=\ &[\left(F^{\upsilon+1}\right)^{H}X^{H},\left(F^{\upsilon+1}\right)^{H}Y^{H}]-\left(F^{\upsilon+1}\right)^{H}[\left(F^{\upsilon+1}\right)^{H}X^{H},Y^{H}]\\
&-\left(F^{\upsilon+1}\right)^{H}[X^{H},\left(F^{\upsilon+1}\right)^{H}Y^{H}]+\left(F^{\upsilon+1}\right)^{H}\left(F^{\upsilon+1}\right)^{H}\left[X^{H},Y^{H}\right]\\
=\ &\lambda^{4}\{\left[F^{\upsilon-1}X+F^{\upsilon-1}Y\right]^{H}-\gamma R\left(F^{\upsilon-1}X,F^{\upsilon-1}Y\right)\\
&-\left(F^{\upsilon-1}\right)^{H}(\left[F^{\upsilon-1}X,Y\right]^{H}+\gamma R\left(F^{\upsilon-1}X,Y\right))\\
&-\left(F^{\upsilon-1}\right)^{H}(\left[X,F^{\upsilon-1}Y\right]^{H}+\gamma R\left(X,F^{\upsilon-1}Y\right))\\
&+(\left(F^{\upsilon-1}\right)^{2}[X,Y])^{H}+\gamma R\left(X,Y\right)\left(F^{\upsilon-1}\right)^{2}\}\\
=\ &\lambda^{4}\{(N_{F^{\upsilon-1}F^{\upsilon-1}}\left(X,Y\right))^{H}+\gamma\{R\left(F^{\upsilon-1}X,F^{\upsilon-1}Y\right)\\
&-R\left(F^{\upsilon-1}X,Y\right)F^{\upsilon-1}-R\left(X,F^{\upsilon-1}Y\right)F^{\upsilon-1}+R\left(X,Y\right)\left(F^{\upsilon-1}\right)^{2}\}\}.
\end{aligned}$$

Let us suppose that the curvature tensor R of ∇ satisfies

$$R\left(F^{\upsilon-1}X,F^{\upsilon-1}Y\right)-R\left(F^{\upsilon-1}X,Y\right)F^{\upsilon-1}-R\left(X,F^{\upsilon-1}Y\right)F^{\upsilon-1}+R\left(X,Y\right)\left(F^{\upsilon-1}\right)^{2}=0$$

and the Nijenhuis tensor of the $F^{\upsilon-1}$ is zero. So, we get

$$\tilde{N}_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}\left(X^{H},Y^{H}\right)=0.$$

ii)

$$\begin{aligned}
&\tilde{N}_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}\left(X^{H},\omega^{v}\right)\\
=\ &[\left(F^{\upsilon+1}\right)^{H}X^{H},\left(F^{\upsilon+1}\right)^{H}\omega^{V}]-\left(F^{\upsilon+1}\right)^{H}[\left(F^{\upsilon+1}\right)^{H}X^{H},\omega^{v}]\\
&-\left(F^{\upsilon+1}\right)^{H}[X^{H},\left(F^{\upsilon+1}\right)^{H}\omega^{v}]+\left(F^{\upsilon+1}\right)^{H}\left(F^{\upsilon+1}\right)^{H}\left[X^{H},\omega^{v}\right]\\
=\ &\lambda^{4}\{[\left(F^{\upsilon-1}X\right)^{H}+\left(\omega\circ F^{\upsilon-1}\right)^{v}]-\left(F^{\upsilon-1}\right)^{H}[\left(F^{\upsilon-1}X\right)^{H},\omega^{v}]\\
&-\left(F^{\upsilon-1}\right)[X^{H},\left(\omega\circ F^{\upsilon-1}\right)^{v}]+(\left(F^{\upsilon-1}\right)^{2})^{H}\left(\nabla_{X}\omega\right)^{v}\}\\
=\ &\lambda^{4}\{\left(\nabla_{F^{\upsilon-1}X}\left(\omega\circ F^{\upsilon-1}\right)\right)^{v}-\left(\left(\nabla_{F^{\upsilon-1}X}\omega\right)F^{\upsilon-1}\right)^{v}\\
&-\left(\left(\nabla_{X}\left(\omega\circ F^{\upsilon-1}\right)\right)F^{\upsilon-1}\right)^{v}+((\nabla_{X}\omega)\left(F^{\upsilon-1}\right)^{2})^{v}\}\\
=\ &\lambda^{4}\{\left(\omega\left(\nabla_{F^{\upsilon-1}X}F^{\upsilon-1}\right)\right)^{v}-\left(\left(\omega\left(\nabla_{X}F^{\upsilon-1}\right)\right)F^{\upsilon-1}\right)^{v}\}.
\end{aligned}$$

We now suppose $\nabla F^{\upsilon-1}=0$, then we see $\tilde{N}_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}\left(X^{H},\omega^{v}\right)=0$.

iii)

$$\begin{aligned}
&\tilde{N}_{(F^{\upsilon+1})(F^{\upsilon+1})}\left(\omega^{v},\theta^{v}\right)\\
=\ &\left[\left(F^{\upsilon+1}\right)\omega^{v},(F^{\upsilon+1})\theta^{v}\right]-\left(F^{\upsilon+1}\right)\left[(F^{\upsilon+1})\omega^{v},\theta^{v}\right]\\
&-\left(F^{\upsilon+1}\right)\left[\omega^{v},\left(F^{\upsilon+1}\right)\theta^{v}\right]+\left(F^{\upsilon+1}\right)\left(F^{\upsilon+1}\right)\left[\omega^{v},\theta^{v}\right]\\
=\ &\lambda^{4}\{[\left(\omega\circ F^{\upsilon-1}\right)^{v},\left(\theta\circ F^{\upsilon-1}\right)^{v}]-F^{\upsilon-1}[\left(\omega\circ F^{\upsilon-1}\right)^{v},\theta^{v}]
\end{aligned}$$

$$-\left(F^{\upsilon-1}\right)[\omega^v,\left(\theta\circ F^{\upsilon-1}\right)^v]+\left(F^{\upsilon-1}\right)^2[\omega^v,\theta^v]=0.$$

Because $[\omega^v,\theta^v]=0$ and $\omega\circ F^{\upsilon-1}\in\Im_1^0(M^n)$ on $T^*(M^n)$, the Nijenhuis tensor $\tilde{N}_{(F^{\upsilon+1})^H,(F^{\upsilon+1})^H}(\omega^v,\theta^v)$ of the horizontal lift $F^{\upsilon+1}$ vanishes. □

Proposition 14. *Let $(F^{\upsilon+1})^H$ be a tensor field of type $(1,1)$ on $T^*(M^n)$. If the Tachibana operator $\phi_{(F^{\upsilon+1})^H}$ applied to vector and covector fields according to horizontal lifts of $F^{\upsilon+1}$ defined by (20) on $T^*(M^n)$, then we get the following results.*

$$\begin{aligned} i)\ \phi_{(F^{\upsilon+1})^H X^H}Y^H &= \lambda^2\{-((L_YF^{\upsilon-1})X)^H-(pR(Y,F^{\upsilon-1}X))^v\\ &\quad+((pR(Y,X))F^{\upsilon-1})^v\},\\ ii)\ \phi_{(F^{\upsilon+1})^H X^H}\omega^v &= \lambda^2\{(\nabla_{F^{\upsilon-1}X}\omega)^v-((\nabla_X\omega)\circ F^{\upsilon-1})^v\},\\ iii)\ \phi_{(F^{\upsilon+1})^H\omega^V}X^H &= -\lambda^2(\omega\circ(\nabla_XF^{\upsilon-1}))^v,\\ iv)\ \phi_{(F^{\upsilon+1})^H\omega^V}\theta^v &= 0,\end{aligned}$$

where horizontal lifts $X^H,Y^H\in\Im_0^1(T^*(M^n))$ of $X,Y\in\Im_0^1(M^n)$ and the vertical lift $\omega^v,\theta^v\in\Im_0^1(T^*(M^n))$ of $\omega,\theta\in\Im_1^0(M^n)$ are given, respectively.

Proof. *i)*

$$\begin{aligned}\phi_{(F^{\upsilon+1})^H X^H}Y^H &= -(L_{Y^H}(F^{\upsilon+1})^H)X^H\\ &= -L_{Y^H}(F^{\upsilon+1})^HX^H+(F^{\upsilon+1})^HL_{Y^H}X^H\\ &= \lambda^2\{-((L_YF^{\upsilon-1})X)^H-(pR(Y,F^{\upsilon-1}X))^v\\ &\quad+((pR(Y,X))F^{\upsilon-1})^v\}\end{aligned}$$

ii)

$$\begin{aligned}\phi_{(F^{\upsilon+1})^H X^H}\omega^v &= -(L_{\omega^V}(F^{\upsilon+1})^H)X^H\\ &= -L_{\omega^V}(F^{\upsilon+1})^HX^H+(F^{\upsilon+1})^HL_{\omega^V}X^H\\ &= -\lambda^2L_{\omega^V}(F^{\upsilon-1}X)^H-\lambda^2(F^{\upsilon-1})^H(\nabla_X\omega)^v\\ &= \lambda^2\{(\nabla_{F^{\upsilon-1}X}\omega)^V-((\nabla_X\omega)\circ F^{\upsilon-1})^V\},\end{aligned}$$

iii)

$$\begin{aligned}\phi_{(F^{\upsilon+1})^H\omega^V}X^H &= -(L_{X^H}(F^{\upsilon+1})^H)\omega^v\\ &= -\lambda^2(\nabla_X(\omega\circ F^{\upsilon-1}))^v+\lambda^2((\nabla_X\omega)\circ F^{\upsilon-1})^v\\ &= -\lambda^2(\omega\circ(\nabla_XF^{\upsilon-1}))^v\end{aligned}$$

iv)

$$\phi_{(F^{\upsilon+1})^H\omega^V}\theta^v = -(L_{\theta^V}(F^{\upsilon+1})^H)\omega^v$$

$$\begin{aligned} &= -L_{\theta^V}(F^{\upsilon+1})^H\omega^v + (F^{\upsilon+1})^H L_{\theta^V}\omega^v \\ &= 0 \end{aligned}$$

□

Proposition 15. *The horizontal lift Y^H is an holomorfic vector field with respect to the structure $\left(F^{\upsilon+1}\right)^H - \lambda^2\left(F^{\upsilon-1}\right)^H = 0$, If $L_Y F^{\upsilon-1} = 0$ and $R\left(Y, F^{\upsilon-1}X\right) = -R(Y,X)F^{\upsilon-1}$.*

Proof. i)

$$\begin{aligned} (L_{Y^H}\left(F^{\upsilon+1}\right)^H)X^H &= L_{Y^H}\left(F^{\upsilon+1}\right)^H X^H - \left(F^{\upsilon+1}\right)^H L_{Y^H}X^H \\ &= \lambda^2(\left[Y, F^{\upsilon-1}X\right]^H + \gamma R\left(Y, F^{\upsilon-1}X\right)) \\ &\quad -\lambda^2(\left(F^{\upsilon-1}[Y,X]\right)^H + \lambda R(Y,X)F^{\upsilon-1}) \\ &= \lambda^2\{\left(\left(L_Y F^{\upsilon-1}\right)X\right)^H + \gamma\{R\left(Y, F^{\upsilon-1}X\right) \\ &\quad -R(Y,X)F^{\upsilon-1}\}\} \end{aligned}$$

ii)

$$\begin{aligned} (L_{X^H}\left(F^{\upsilon+1}\right)^H)\omega^v &= L_{X^H}\left(F^{\upsilon+1}\right)^H\omega^v - \left(F^{\upsilon+1}\right)^H L_{X^H}\omega^v \\ &= \lambda^2\left(\nabla_X\left(\omega\circ F^{\upsilon-1}\right)\right)^v - \lambda^2\left(\left(\nabla_X\omega\right)F^{\upsilon-1}\right)^v \\ &= \lambda^2\{\left(\nabla_X\left(\omega\circ F^{\upsilon-1}\right)\right)^v - \left(\left(\nabla_X\omega\right)F^{\upsilon-1}\right)^v\} \end{aligned}$$

□

Theorem 16. *Let $(T^*(M^n),{}^S g)$ be the cotangent bundle equipped with Sasakian metric ${}^S g$ and a tensor field $(F^{\upsilon+1})$ of type $(1,1)$ defined by (1). Sasakian metric ${}^S g$ is pure with respect to $(F^{\upsilon+1})^H$ if $F^{\upsilon-1} = \lambda^2 I$. ($I$=Identity tensor field of type $(1,1)$).*

Proof. We put

$$S(\tilde{X},\tilde{Y}) = {}^S g((F^{\upsilon+1})^H\tilde{X},\tilde{Y}) - {}^S g(\tilde{X},(F^{\upsilon+1})^H\tilde{Y}).$$

If $S(\tilde{X},\tilde{Y}) = 0$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ω^v, θ^v or X^H, Y^H, then $S = 0$. By virtue of $(F^{\upsilon+1})^H - \lambda^2(F^{\upsilon-1})^H = 0$ and (26),(27), (28), we get

i)

$$\begin{aligned} S(\omega^v,\theta^v) &= {}^S g(\left(F^{\upsilon+1}\right)^H\omega^v,\theta^v) - {}^S g(\omega^v,\left(F^{\upsilon+1}\right)^H\theta^v) \\ &= \lambda^2\{{}^S g(\left(\omega\circ F^{\upsilon-1}\right)^v,\theta^v) - {}^S g(\omega^v,\left(\theta\circ F^{\upsilon-1}\right)^v)\} \\ &= \lambda^2\{\left(g^{-1}(\left(\omega\circ F^{\upsilon-1}\right),\theta)\right)^v - \left(g^{-1}\left(\omega,\left(\theta\circ F^{\upsilon-1}\right)\right)\right)^v\} \end{aligned}$$

ii)

$$S\left(X^H,\theta^v\right) = {}^S g(\left(F^{\upsilon+1}\right)^H X^H,\theta^v) - {}^S g(X^H,\left(F^{\upsilon+1}\right)^H\theta^v)$$

$$= \lambda^2\{^S g(\left(F^{\upsilon-1}X\right)^H, \theta^v) -^S g(X^H, \left(\omega \circ F^{\upsilon-1}\right)^v)$$
$$= 0$$

iii)

$$\begin{aligned} S\left(X^H, Y^H\right) &= {}^S g(\left(F^{\upsilon+1}\right)^H X^H, Y^H) -^S g(X^H, \left(F^{\upsilon+1}\right)^H Y^H) \\ &= \lambda^2\{^S g(\left(F^{\upsilon-1}X\right)^H, Y^H) -^S g(X^H, \left(F^{\upsilon-1}Y\right)^H)\} \\ &= \lambda^2\{\left(g\left(\left(F^{\upsilon-1}X\right), Y\right)\right)^v - \left(g\left(X, \left(F^{\upsilon-1}Y\right)\right)\right)^v\} \end{aligned}$$

We now suppose $F^{\upsilon-1} = \lambda^2 I$, then we get ${}^S g = 0$. So, ${}^S g$ is pure with respect to $(F^{\upsilon+1})^H$. □

2.5. **The structure** $(F^{\upsilon+1})^C - \lambda^2(F^{\upsilon-1})^C = 0$ **on tangent bundle** $T\,(M^n)$**.** Let M^n be an n-dimensional connected differentiable manifold of class C^∞. Let there be given in M^n, a $(1,1)$ tensor field F of class C^∞ satisfying [14, 21]

$$F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0, \tag{30}$$

where λ is non zero complex number. Also rank (F)

$$= \frac{1}{2}\left(rank\ F^{\upsilon+1} + \dim\ M^n\right)$$
$$= r\,(a\ cons\tan t\ every\ where\ on\ M^n)$$

Let the operators l^* and m^* be defined as

$$l^* def\,(F/\lambda)^{\upsilon-1}, m^* = I - (F/\lambda)^{\upsilon-1},$$

where I denotes the identity operator on M^n. Then the operators I^* and m^* applied to the tangent space at a point of the manifold be complementary projection operators.

Let F_i^h be the component of F at A in the coordinate neighbourhood U of M^n. Then the complete lift F^C of F is also a tensor field of type $(1,1)$ in $T(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U) : M^n \longrightarrow T(M^n)$ are given by [25]

$$F^C = \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}. \tag{31}$$

Let $F, G \in \Im_1^1(M^n)$ then we have

$$(FG)^C = F^C G^C. \tag{32}$$

Putting $F = G$ we obtain

$$\left(F^2\right)^C = \left(F^C\right)^2. \tag{33}$$

Putting $G = F^2$ in (32) and making use of (33) we get

$$\left(F^3\right)^C = \left(F^C\right)^3. \tag{34}$$

Continuing the above process of replacing G in equation (32) by some higher degree of F we obtain

$$\left(F^{\upsilon+1}\right)^C=\left(F^C\right)^{\upsilon+1}. \tag{35}$$

Taking complete lift on both sides of equation (30) we get

$$\left(F^{\upsilon+1}\right)^C-\lambda^2(F^{\upsilon-1})^C=0 \tag{36}$$

which in view of the equation (35) gives

$$\left(F^C\right)^{\upsilon+1}-\lambda^2\left(F^C\right)^{\upsilon-1}=0. \tag{37}$$

The complete lift of a $F^{\upsilon+1}-\lambda^2F^{\upsilon-1}=0$ structure also has $F^{\upsilon+1}-\lambda^2F^{\upsilon-1}=0$ structure in tangent bundle.

Lemma 17. *Let X and Y be any vector fields on a Riemannian manifold (M^n,g), we have [25]*

$$\begin{aligned}\left[X^H,Y^H\right] &= [X,Y]^H-\left(R\left(X,Y\right)u\right)^v,\\ \left[X^H,Y^v\right] &= \left(\nabla_XY\right)^v,\\ \left[X^v,Y^v\right] &= 0,\end{aligned}$$

where R is the Riemannian curvature tensor of g defined by

$$R\left(X,Y\right)=\left[\nabla_X,\nabla_Y\right]-\nabla_{[X,Y]}.$$

In particular, we have the vertical spray u^v and the horizontal spray u^H on $T(M^n)$ defined by

$$u^V=u^i\left(\partial_i\right)^v=u^i\partial_{\bar{i}},\ u^H=u^i\left(\partial_i\right)^H=u^i\delta_i,$$

where $\delta_i=\partial_i-u^j\Gamma^s_{ji}\partial_{\bar{s}}$. u^v is also called the canonical or Liouville vector field on $T(M^n)$.

Theorem 18. *The Nijenhuis tensor $N_{(F^{\upsilon+1})^C(F^{\upsilon+1})^C}\left(X^C,Y^C\right)$ of the complete lift of $F^{\upsilon+1}$ vanishes if the Nijenhuis tensor of the $F^{\upsilon-1}$ is zero.*

Proof. In consequence of Definition 2 the Nijenhuis tensor of $\left(F^{\upsilon+1}\right)^C$ is given by

$$\begin{aligned}&N_{(F^{\upsilon+1})^C(F^{\upsilon+1})^C}\left(X^C,Y^C\right)\\ =\;&[\left(F^{\upsilon+1}\right)^CX^C,\left(F^{\upsilon+1}\right)^CY^C]-\left(F^{\upsilon+1}\right)^C[\left(F^{\upsilon+1}\right)^CX^C,Y^C]\\ &-\left(F^{\upsilon+1}\right)^C[X^C,\left(F^{\upsilon+1}\right)^CY^C]+\left(F^{\upsilon+1}\right)^C\left(F^{\upsilon+1}\right)^C\left[X^C,Y^C\right]\\ =\;&\lambda^4\{[\left(F^{\upsilon-1}X\right)^C,\left(F^{\upsilon-1}Y\right)^C]-\left(F^{\upsilon-1}\right)^C[\left(F^{\upsilon-1}X\right)^C,Y^C]\\ &-\left(F^{\upsilon-1}\right)^C[X^C,\left(F^{\upsilon-1}Y\right)^C]+\left(F^{\upsilon-1}\right)^C\left(F^{\upsilon-1}\right)^C\left[X^C,Y^C\right]\}\\ =\;&\lambda^4\{\left[F^{\upsilon-1}X,F^{\upsilon-1}Y\right]-F^{\upsilon-1}\left[F^{\upsilon-1}X,Y\right]\\ &-F^{\upsilon-1}\left[X,F^{\upsilon-1}Y\right]+F^{\upsilon-1}F^{\upsilon-1}\left[X,Y\right]\}^C\\ =\;&\lambda^4N_{F^{\upsilon-1}F^{\upsilon-1}}\left(X,Y\right)^C\end{aligned}$$

□

Theorem 19. *The Nijenhuis tensor* $N_{(F^{v+1})^C(F^{v+1})^C}\left(X^C,Y^V\right)$ *of the complete lift of* F^{v+1} *vanishes if the Nijenhius tensor* F^{v-1} *is zero.*

Proof.

$$\begin{aligned} & N_{(F^{v+1})^C(F^{v+1})^C}\left(X^C,Y^v\right) \\ = \ & [\left(F^{v+1}\right)^C X^C,\left(F^{v+1}\right)^C Y^v]-\left(F^{v+1}\right)^C[\left(F^{v+1}\right)^C X^C,Y^v] \\ & -\left(F^{v+1}\right)^C[X^C,\left(F^{v+1}\right)^C Y^v]+\left(F^{v+1}\right)^C\left(F^{v+1}\right)^C\left[X^C,Y^v\right] \\ = \ & \lambda^4\{[\left(F^{v-1}X\right)^C,\left(F^{v-1}Y\right)^v]-\left(F^{v-1}\right)^C[\left(F^{v-1}X\right)^C,Y^v] \\ & -\left(F^{v-1}\right)^C[X^C,\left(F^{v-1}Y\right)^v]+\left(F^{v-1}\right)^C\left(F^{v-1}\right)^C[X,Y]^v\} \\ = \ & \lambda^4\{\left[F^{v-1}X,F^{v-1}Y\right]^v-\left(F^{v-1}\left[F^{v-1}X,Y\right]\right)^v \\ & -\left(F^{v-1}\left[X,F^{v-1}Y\right]\right)^v-\left(F^{v-1}F^{v-1}[X,Y]\right)^v\} \\ = \ & \lambda^4 N_{F^{v-1}F^{v-1}}(X,Y)^v \end{aligned}$$

□

Theorem 20. *The Nijenhuis tensor* $N_{(F^{v+1})^C(F^{v+1})^C}(X^v,Y^v)$ *of the complete lift of* F^{v+1} *vanishes.*

Proof. Because $[X^v,Y^v]=0$ and $F^{v-1}X\in\Im_0^1(M^n)$, easily we get the Nijenhuis tensor $N_{(F^{v+1})^C(F^{v+1})^C}(X^v,Y^v)=0$. □

2.6. **The purity conditions of Sasakian metric with respect to** $(F^{v+1})^C$ **on** $T(M^n)$**.**

Definition 21. *The Sasaki metric* ${}^Sg$ *is a (positive definite) Riemannian metric on the tangent bundle* $T(M^n)$ *which is derived from the given Riemannian metric on* M *as follows:*

$$\begin{aligned} {}^Sg\left(X^H,Y^H\right) &= g(X,Y), \\ {}^Sg\left(X^H,Y^v\right) &= {}^Sg\left(X^v,Y^H\right)=0, \\ {}^Sg\left(X^v,Y^v\right) &= g(X,Y) \end{aligned} \tag{38}$$

for all $X,Y\in\Im_0^1(M^n)$ [20].

Theorem 22. *The Sasaki metric* ${}^Sg$ *is pure with respect to* $\left(F^{v+1}\right)^C$ *if* $\nabla F^{v-1}=0$ *and* $F^{v-1}=\lambda^2 I$ *, where I=ıdentity tensor field of type* $(1,1)$*.*

Proof. $S(\widetilde{X},\widetilde{Y})={}^S g(\left(F^{v+1}\right)^C\widetilde{X},\widetilde{Y})-{}^S g(\widetilde{X},\left(F^{v+1}\right)^C\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y})=0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^v,Y^v or X^H,Y^H then $S=0$.

i)

$$S(X^v,Y^v) = {}^Sg(\left(F^{v+1}\right)^C X^v,Y^v)-{}^S g(X^v,\left(F^{v+1}\right)^C Y^v)$$

$$\begin{aligned} &= \lambda^2\{{}^S g(\left(F^{\upsilon-1}X\right)^{\upsilon},Y^{\upsilon}) -{}^S g(X^{\upsilon},\left(F^{\upsilon-1}Y\right)^{\upsilon})\} \\ &= \lambda^2\{\left(g\left(F^{\upsilon-1}X,Y\right)\right)^{\upsilon} - \left(g\left(X,F^{\upsilon-1}Y\right)\right)^{\upsilon}\} \end{aligned}$$

ii)

$$\begin{aligned} S\left(X^{\upsilon},Y^{H}\right) &= {}^S g(\left(F^{\upsilon+1}\right)^{C}X^{\upsilon},Y^{H}) -{}^S g(X^{\upsilon},\left(F^{\upsilon+1}\right)^{C}Y^{H}) \\ &= -\lambda^2\ {}^S g(X^{\upsilon},\left(F^{\upsilon-1}Y\right)^{H} + \left(\nabla_{\gamma}F^{\upsilon-1}\right)Y^{H}) \\ &= -\lambda^2\ {}^S g(X^{\upsilon},\left(\left(\left(\nabla F^{\upsilon-1}\right)u\right)Y\right)^{\upsilon}) \\ &= -\lambda^2(\ g\left(X,\left(\left(\nabla F^{\upsilon-1}\right)u\right)Y\right)^{\upsilon}) \end{aligned}$$

iii)

$$\begin{aligned} S\left(X^{H},Y^{H}\right) &= {}^S g(\left(F^{\upsilon+1}\right)^{C}X^{H},Y^{H}) -{}^S g(X^{H},\left(F^{\upsilon+1}\right)^{C}Y^{H}) \\ &= \lambda^2\ {}^S g(\left(F^{\upsilon-1}\right)^{C}X^{H},Y^{H}) - \lambda^2\ {}^S g(X^{H},\left(F^{\upsilon-1}\right)^{C}Y^{H}) \\ &= \lambda^2\ {}^S g(\left(F^{\upsilon-1}X\right)^{H} + \left(\nabla_{\gamma}F^{\upsilon-1}\right)X^{H},Y^{H}) \\ &\quad -\lambda^2\ {}^S g(X^{H},\left(F^{\upsilon-1}Y\right)^{H} + \left(\nabla_{\gamma}F^{\upsilon-1}\right)Y^{H}) \\ &= \lambda^2\{g\left(\left(F^{\upsilon-1}X\right),Y\right)^{\upsilon} - g\left(X,\left(F^{\upsilon-1}Y\right)\right)^{\upsilon}\} \end{aligned}$$

We now suppose $\nabla F^{\upsilon-1}=0$ and $F^{\upsilon-1}=\lambda^2 I$, then we get ${}^S g=0$. So, ${}^S g$ is pure with respect to $(F^{\upsilon+1})^C$. □

Theorem 23. *Let $(F^{\upsilon+1})^C$ be a tensor field of type $(1,1)$ on $T(M^n)$. If the Tachibana operator $\phi_{(F^{\upsilon+1})^C}$ applied to vector fields according to complete lifts of $F^{\upsilon+1}$ defined by (36) on $T(M^n)$, then we get the following results.*

$$\begin{aligned} i)\ \phi_{(F^{\upsilon+1})^C X^C}Y^C &= -\lambda^2\left(\left(L_Y F^{\upsilon-1}\right)X\right)^C, \\ ii)\ \phi_{(F^{\upsilon+1})^C X^C}Y^{\upsilon} &= -\lambda^2\left(\left(L_Y F^{\upsilon-1}\right)X\right)^{\upsilon}, \\ iii)\ \phi_{(F^{\upsilon+1})^C X^V}Y^C &= -\lambda^2\left(\left(L_Y F^{\upsilon-1}\right)X\right)^{\upsilon}, \\ iv)\ \phi_{(F^{\upsilon+1})^C X^{\upsilon}}Y^{\upsilon} &= 0, \end{aligned}$$

where $X,Y\in\Im_0^1(M)$, the complete lifts $X^C,Y^C\in\Im_0^1(T(M))$ and the vertical lift $X^{\upsilon},Y^{\upsilon}\in\Im_0^1(T(M))$.

Proof. i)

$$\begin{aligned} \phi_{(F^{\upsilon+1})^C X^C}Y^C &= -(L_{Y^C}\left(F^{\upsilon+1}\right)^C)X^C \\ &= \lambda^2\{-L_{Y^C}\left(F^{\upsilon-1}X\right)^C + \left(F^{\upsilon-1}\right)^C L_{Y^C}X^C\} \\ &= -\lambda^2\left(\left(L_Y F^{\upsilon-1}\right)X\right)^C \end{aligned}$$

ii)

$$\phi_{(F^{\upsilon+1})^C X^C}Y^{\upsilon} = -(L_{Y^V}\left(F^{\upsilon+1}\right)^C)X^C$$

$$
\begin{aligned}
&= -L_{Y^V}\left(F^{v+1}\right)^C X^C + \left(F^{v+1}\right)^C L_{Y^V} X^C \\
&= \lambda^2\{-L_{Y^V}\left(F^{v-1}X\right)^C + \left(F^{v-1}\right)^C L_{Y^V} X^C\} \\
&= -\lambda^2\left(\left(L_Y F^{v-1}\right)X\right)^v
\end{aligned}
$$

iii)

$$
\begin{aligned}
\phi_{(F^{v+1})^C X^v} Y^C &= -(L_{Y^C}\left(F^{v+1}\right)^C)X^v \\
&= -L_{Y^C}\left(F^{v+1}\right)^C X^v + \left(F^{v+1}\right)^C L_{Y^C} X^v \\
&= \lambda^2\{-L_{Y^C}\left(F^{v-1}X\right)^v + \left(F^{v-1}\right)^C L_{Y^C} X^v\} \\
&= -\lambda^2\left(\left(L_Y F^{v-1}\right)X\right)^v
\end{aligned}
$$

iv)

$$
\begin{aligned}
\phi_{(F^{v+1})^C X^v} Y^v &= -(L_{Y^V}\left(F^{v+1}\right)^C)X^v \\
&= -L_{Y^V}\left(F^{v+1}\right)^C X^v + \left(F^{v+1}\right)^C L_{Y^V} X^v \\
&= 0
\end{aligned}
$$

□

Theorem 24. *The complete lift Y^C is an holomorfic vector field with respect to the structure $\left(F^{v+1}\right)^C - \lambda^2\left(F^{v-1}\right)^C = 0$, If $L_Y F^{v-1} = 0$.*

Proof. i)

$$
\begin{aligned}
(L_{Y^C}\left(F^{v+1}\right)^C)X^C &= L_{Y^C}\left(F^{v+1}\right)^C X^C - \left(F^{v+1}\right)^C L_{Y^C} X^C \\
&= \lambda^2\{L_{Y^C}\left(F^{v-1}X\right)^C - \left(F^{v-1}\right)^C L_{Y^C} X^C\} \\
&= \lambda^2\left(\left(L_Y F^{v-1}\right)X\right)^C
\end{aligned}
$$

ii)

$$
\begin{aligned}
(L_{Y^C}\left(F^{v+1}\right)^C)X^v &= L_{Y^C}\left(F^{v+1}\right)^C X^v - \left(F^{v+1}\right)^C L_{Y^C} X^v \\
&= \lambda^2\{L_{Y^C}\left(F^{v-1}X\right)^v - \left(F^{v-1}\right)^C L_{Y^C} X^v\} \\
&= \lambda^2\left(\left(L_Y F^{v-1}\right)X\right)^v
\end{aligned}
$$

□

2.7. **The structure $\left(F^{v+1}\right)^H - \lambda^2\left(F^{v-1}\right)^H = 0$ on tangent bundle $T(M^n)$.** Let F_i^h be the component of F at A in the coordinate neighbourhood U of M^n. Then the horizontal lift F^H of F is also a tensor field of type $(1,1)$ in $T(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U) : M^n \longrightarrow T(M^n)$ are given by [25]

$$
F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F_i^h & 0 \\ -\Gamma_t^h F_i^t + \Gamma_i^t F_t^h & F_i^h \end{pmatrix}. \tag{39}
$$

Let F, G be two tensor fields of type $(1,1)$ on the manifold M. If F^H denotes the horizontal lift of F, we have

$$(FG)^H = F^H G^H \tag{40}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H \tag{41}$$

Multiplying both sides by F^H and making use of the same (41), we get

$$(F^H)^3 = (F^3)^H.$$

Thus it follows that

$$(F^H)^{\upsilon+1} = (F^{\upsilon+1})^H \tag{42}$$

Taking horizontal lift on both sides of equation $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0$ we get

$$(F^{\upsilon+1})^H - \lambda^2 (F^{\upsilon-1})^H = 0 \tag{43}$$

In view of (42), we can write

$$(F^H)^{\upsilon+1} - \lambda^2 \left(F^H\right)^{\upsilon-1} = 0. \tag{44}$$

Thus the horizontal lift of $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0$ structure also has $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0$ structure in tangent bundle $T(M^n)$.

Theorem 25. *The Nijenhuis tensor $N_{(F^{\upsilon+1})^H (F^{\upsilon+1})^H}\left(X^H, Y^H\right)$ of the horizontal lift of $F^{\upsilon+1}$ vanishes if the Nijenhuis tensor of the $F^{\upsilon-1}$ is zero and*

$$\{-(\hat{R}\left(F^{\upsilon-1}X, F^{\upsilon-1}Y\right)u) + (F^{\upsilon-1}(\hat{R}\left(F^{\upsilon-1}X, Y\right)u))$$

$$+\left(F^{\upsilon-1}\left(R\left(X, F^{\upsilon-1}Y\right)u\right)\right) - (\left(F^{\upsilon-1}\right)^2 (\hat{R}\left(X, Y\right)u))\}^{\upsilon} = 0$$

Proof.

$$\begin{aligned}
& N_{(F^{\upsilon+1})^H (F^{\upsilon+1})^H}\left(X^H, Y^H\right) \\
= \; & [\left(F^{\upsilon+1}\right)^H X^H, \left(F^{\upsilon+1}\right)^H Y^H] \\
& - \left(F^{\upsilon+1}\right)^H [\left(F^{\upsilon+1}\right)^H X^H, Y^H] \\
& - \left(F^{\upsilon+1}\right)^H [X^H, \left(F^{\upsilon+1}\right)^H Y^H] \\
& + \left(F^{\upsilon+1}\right)^H \left(F^{\upsilon+1}\right)^H \left[X^H, Y^H\right] \\
= \; & \lambda^4 \{(\left[F^{\upsilon-1}X, F^{\upsilon-1}Y\right] - \left(F^{\upsilon-1}\right)\left[F^{\upsilon-1}X, Y\right] \\
& - \left(F^{\upsilon-1}\right)\left[X, F^{\upsilon-1}Y\right] - \left(F^{\upsilon-1}\right)\left(F^{\upsilon-1}\right)[X, Y])^H \\
& -(\hat{R}\left(F^{\upsilon-1}X, F^{\upsilon-1}Y\right)u)^{\upsilon} + (F^{\upsilon-1}(\hat{R}\left(F^{\upsilon-1}X, Y\right)u))^{\upsilon} \\
& +(F^{\upsilon-1}(\hat{R}\left(X, F^{\upsilon-1}Y\right)u))^{\upsilon} - (\left(F^{\upsilon-1}\right)^2 (\hat{R}\left(X, Y\right))u)^{\upsilon}\} \\
= \; & \lambda^4 \{(N_{F^{\upsilon-1}F^{\upsilon-1}}\left(X, Y\right))^H - (\hat{R}\left(F^{\upsilon-1}X, F^{\upsilon-1}Y\right)u)^{\upsilon}
\end{aligned}$$

$$+(F^{\upsilon-1}(\hat{R}\left(F^{\upsilon-1}X,Y\right)u))^{v}+(F^{\upsilon-1}(\hat{R}\left(X,F^{\upsilon-1}Y\right)u))^{v}$$
$$-(\left(F^{\upsilon-1}\right)^{2}(\hat{R}\left(X,Y\right)u))^{v}\}.$$

□

If $N_{F^{\upsilon-1}F^{\upsilon-1}}(X,Y)=0$ and $\{-\hat{R}\left(F^{\upsilon-1}X,F^{\upsilon-1}Y\right)u+(F^{\upsilon-1}(\hat{R}\left(F^{\upsilon-1}X,Y\right)u))+(F^{\upsilon-1}(\hat{R}\left(X,F^{\upsilon-1}Y\right)u))-(\left(F^{\upsilon-1}\right)^{2}(\hat{R}\left(X,Y\right)u))\}^{v}=0$,
then we get $N_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}\left(X^{H},Y^{H}\right)=0$. The theorem is proved.
Where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$ defined by $\hat{\nabla}_{X}Y=\nabla_{Y}X+[X,Y]$ (see [25] p.88-89).

Theorem 26. *The Nijenhuis tensor $N_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}\left(X^{H},Y^{v}\right)$ of the horizontal lift of $F^{\upsilon+1}$ vanishes if the Nijenhuis tensor of the $F^{\upsilon-1}$ is zero and $\nabla F^{\upsilon-1}=0$.*

Proof.

$$\begin{aligned}
N_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}\left(X^{H},Y^{V}\right) &= [\left(F^{\upsilon+1}\right)^{H}X^{H},\left(F^{\upsilon+1}\right)^{H}Y^{v}] \\
&\quad-\left(F^{\upsilon+1}\right)^{H}[\left(F^{\upsilon+1}\right)^{H}X^{H},Y^{v}] \\
&\quad-\left(F^{\upsilon+1}\right)^{H}[X^{H},\left(F^{\upsilon+1}\right)^{H}Y^{v}] \\
&\quad+\left(F^{\upsilon+1}\right)^{H}\left(F^{\upsilon+1}\right)^{H}\left[X^{H},Y^{v}\right] \\
&= \lambda^{4}\{\left[F^{\upsilon-1}X,F^{\upsilon-1}Y\right]^{v}-\left(F^{\upsilon-1}\left[F^{\upsilon-1}X,Y\right]\right)^{v} \\
&\quad-\left(F^{\upsilon-1}\left[X,F^{\upsilon-1}Y\right]\right)^{v}+(\left(F^{\upsilon-1}\right)^{2}[X,Y])^{v} \\
&\quad+\left(\nabla_{F^{\upsilon-1}Y}F^{\upsilon-1}X\right)^{v}-\left(F^{\upsilon-1}\left(\nabla_{Y}F^{\upsilon-1}X\right)\right)^{v} \\
&\quad-\left(F^{\upsilon-1}\left(\nabla_{F^{\upsilon-1}Y}X\right)\right)^{v}+(\left(F^{\upsilon-1}\right)^{2}\nabla_{Y}X)^{v}\} \\
&= \lambda^{4}\{(N_{F^{\upsilon-1}F^{\upsilon-1}}(X,Y))^{v}+\left(\nabla_{F^{\upsilon-1}Y}F^{\upsilon-1}\right)X \\
&\quad-\left(F^{\upsilon-1}\left(\left(\nabla_{Y}F^{\upsilon-1}\right)X\right)\right)^{v}\}
\end{aligned}$$

□

Theorem 27. *The Nijenhuis tensor $N_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}(X^{v},Y^{v})$ of the horizontal lift of $F^{\upsilon+1}$ vanishes.*

Proof. Because $[X^{v},Y^{v}]=0$ for $X,Y\in M$, we get $N_{(F^{\upsilon+1})^{H}(F^{\upsilon+1})^{H}}(X^{v},Y^{v})=0$. □

Theorem 28. *The Sasakian metric ${}^{S}g$ is pure with respect to $\left(F^{\upsilon+1}\right)^{H}$ if $F^{\upsilon-1}=\lambda^{2}I$, where $I=$identity tensor field of type $(1,1)$.*

Proof. $S(\widetilde{X},\widetilde{Y})={}^{S}g(\left(F^{\upsilon+1}\right)^{H}\widetilde{X},\widetilde{Y})-{}^{S}g(\widetilde{X},\left(F^{\upsilon+1}\right)^{H}\widetilde{Y})$ if $S(\widetilde{X},\widetilde{Y})=0$ for all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form X^{v},Y^{v} or X^{H},Y^{H} then $S=0$.

i)

$$S\left(X^{v},Y^{v}\right) = {}^{S}g(\left(F^{\upsilon+1}\right)^{H}X^{v},Y^{v})-{}^{S}g(X^{v},\left(F^{\upsilon+1}\right)^{H}Y^{v})$$

$$\begin{aligned} &= \lambda^{2}\{{}^{S}g(\left(F^{\upsilon-1}X\right)^{v},Y^{v}) -{}^{S} g(X^{v},\left(F^{\upsilon-1}Y\right)^{v})\} \\ &= \lambda^{2}\{\left(g\left(F^{\upsilon-1}X,Y\right)\right)^{v} - \left(g\left(X,F^{\upsilon-1}Y\right)\right)^{v}\} \end{aligned}$$

ii)

$$\begin{aligned} S\left(X^{v},Y^{H}\right) &= {}^{S}g(\left(F^{\upsilon+1}\right)^{H}X^{v},Y^{H}) -{}^{S} g(X^{v},\left(F^{\upsilon+1}\right)^{H}Y^{H}) \\ &= -\lambda^{2}\ {}^{S}g(X^{v},\left(F^{\upsilon-1}Y\right)^{H}) \\ &= 0 \end{aligned}$$

iii)

$$\begin{aligned} S\left(X^{H},Y^{H}\right) &= {}^{S}g(\left(F^{\upsilon+1}\right)^{H}X^{H},Y^{H}) -{}^{S} g(X^{H},\left(F^{\upsilon+1}\right)^{H}Y^{H}) \\ &= \lambda^{2}\ \{({}^{S}g\left(F^{\upsilon-1}X\right)^{H},Y^{H}) -{}^{S} g(X^{H},\left(F^{\upsilon-1}Y\right)^{H})\} \\ &= \lambda^{2}\ \{\left(g\left(F^{\upsilon-1}X\right),Y\right)^{v} - (g(X,\left(F^{\upsilon-1}Y\right)^{H}))^{v}\} \end{aligned}$$

□

Theorem 29. *Let $(F^{\upsilon+1})^{H}$ be a tensor field of type $(1,1)$ on $T(M^{n})$. If the Tachibana operator $\phi_{(F^{\upsilon+1})^{H}}$ applied to vector fields according to horizontal lifts of $F^{\upsilon+1}$ defined by (43) on $T(M^{n})$, then we get the following results.*

$$\begin{aligned} i)\ \phi_{(F^{\upsilon+1})^{H}X^{H}}Y^{H} &= -\lambda^{2}\{-\left(\left(L_{Y}F^{\upsilon-1}\right)X\right)^{H} + (\hat{R}\left(Y,F^{\upsilon-1}X\right)u)^{v} \\ &\quad -(F^{\upsilon-1}(\hat{R}\left(Y,X\right)u))^{v}\}, \\ ii)\ \phi_{(F^{\upsilon+1})^{H}X^{H}}Y^{v} &= \lambda^{2}\{-\left(\left(L_{Y}F^{\upsilon-1}\right)X\right)^{v} + \left(\left(\nabla_{Y}F^{\upsilon-1}\right)X\right)^{v}\}, \\ iii)\ \phi_{(F^{\upsilon+1})^{H}X^{v}}Y^{H} &= \lambda^{2}\{-\left(\left(L_{Y}F^{\upsilon-1}\right)X\right)^{v} - (\nabla_{F^{\upsilon-1}X}Y)^{v} \\ &\quad + \left(F^{\upsilon-1}\left(\nabla_{X}Y\right)\right)^{v}\}, \\ iv)\ \phi_{(F^{\upsilon+1})^{H}X^{v}}Y^{v} &= 0, \end{aligned}$$

where $X,Y\in\Im_{0}^{1}(M)$, the horizontal lifts $X^{H},Y^{H}\in\Im_{0}^{1}(T(M^{n}))$ and the vertical lift $X^{v},Y^{v}\in\Im_{0}^{1}(T(M^{n}))$

Proof. i)

$$\begin{aligned} \phi_{(F^{\upsilon+1})^{H}X^{H}}Y^{H} &= -(L_{Y^{H}}\left(F^{\upsilon+1}\right)^{H})X^{H} \\ &= -\lambda^{2}\left[Y,F^{\upsilon-1}X\right]^{H} + \lambda^{2}\gamma\hat{R}\left[Y,F^{\upsilon-1}X\right] \\ &\quad +\lambda^{2}\left(F^{\upsilon-1}\left[Y,X\right]\right)^{H} - \lambda^{2}\left(F^{\upsilon-1}\right)^{H}(\hat{R}\left(Y,X\right)u)^{v} \\ &= -\lambda^{2}\{-\left(\left(L_{Y}F^{\upsilon-1}\right)X\right)^{H} + (\hat{R}\left(Y,F^{\upsilon-1}X\right)u)^{v} \\ &\quad -(F^{\upsilon-1}(\hat{R}\left(Y,X\right)u))^{v}\} \end{aligned}$$

ii)

$$\phi_{(F^{\upsilon+1})^{H}X^{H}}Y^{V} = -(L_{Y^{V}}\left(F^{\upsilon+1}\right)^{H})X^{H}$$

$$\begin{aligned} &= -\lambda^2\left[Y,F^{\upsilon-1}X\right]^V+\lambda^2\left(\nabla_Y F^{\upsilon-1}X\right)^V \\ &\quad +\lambda^2\left(F^{\upsilon-1}\left[Y,X\right]\right)^v-\lambda^2\left(F^{\upsilon-1}\left(\nabla_Y X\right)\right)^v \\ &= \lambda^2\{-\left(\left(L_Y F^{\upsilon-1}\right)X\right)^v+\left(\left(\nabla_Y F^{\upsilon-1}\right)X\right)^v\} \end{aligned}$$

iii)

$$\begin{aligned} \phi_{(F^{\upsilon+1})^H X^V}Y^H &= -(L_{Y^H}\left(F^{\upsilon+1}\right)^H)X^V \\ &= \lambda^2\left[Y,F^{\upsilon-1}X\right]^v-\lambda^2\left(\nabla_{F^{\upsilon-1}X}Y\right)^v \\ &\quad +\lambda^2\left(F^{\upsilon-1}\left[Y,X\right]\right)^H+\lambda^2\left(F^{\upsilon-1}\left(\nabla_X Y\right)\right)^v \\ &= \lambda^2\{-\left(\left(L_Y F^{\upsilon-1}\right)X\right)^v-\left(\nabla_{F^{\upsilon-1}X}Y\right)^v+\left(F^{\upsilon-1}\left(\nabla_X Y\right)\right)^v\} \end{aligned}$$

iv)

$$\begin{aligned} \phi_{(F^{\upsilon+1})^H X^V}Y^v &= -(L_{Y^V}\left(F^{\upsilon+1}\right)^H)X^v \\ &= -\lambda^2 L_{Y^v}\left(F^{\upsilon-1}X\right)^v+\lambda^2\left(F^{\upsilon-1}\right)^H L_{Y^v}X^v \\ &= 0 \end{aligned}$$

□

Theorem 30. *The horizontal lift Y^H is an holomorfic vector field with respect to $\left(F^{\upsilon+1}\right)^H$, if If $L_Y F^{\upsilon-1}=0$ and $F^{\upsilon-1}=\lambda^2 I$ for $Y\in M$.*

Proof. i)

$$\begin{aligned} (L_{Y^H}\left(F^{\upsilon+1}\right)^H)X^H &= L_{Y^H}\left(F^{\upsilon+1}\right)^H X^H-\left(F^{\upsilon+1}\right)^H L_{Y^H}X^H \\ &= \lambda^2\left[Y,F^{\upsilon-1}X\right]^H-\lambda^2\gamma\hat{R}\left(Y,F^{\upsilon-1}X\right) \\ &\quad -\lambda^2\left(F^{\upsilon-1}\left[Y,X\right]\right)^H+\lambda^2\left(F^{\upsilon-1}\right)^H\left(\hat{R}\left(Y,X\right)u\right)^v \\ &= \lambda^2\{\left(\left(L_Y F^{\upsilon-1}\right)X\right)^H-\left(\hat{R}\left(Y,F^{\upsilon-1}X\right)u\right)^v \\ &\quad +(F^{\upsilon-1}(\hat{R}\left(Y,X\right)u))^v\} \end{aligned}$$

ii)

$$\begin{aligned} (L_{Y^H}\left(F^{\upsilon+1}\right)^H)X^v &= L_{Y^H}\left(F^{\upsilon+1}X\right)^v-\left(F^{\upsilon+1}\right)^H L_{Y^H}X^v \\ &= \lambda^2\left[Y,F^{\upsilon-1}X\right]^v-\lambda^2\left(\nabla_{F^{\upsilon-1}X}Y\right)^v-\lambda^2\left(F^{\upsilon-1}\left[Y,X\right]\right)^v \\ &\quad -\lambda^2\left(F^{\upsilon-1}\left(\nabla_X Y\right)\right)^v \\ &= \lambda^2\{\left(\left(L_Y F^{\upsilon-1}\right)X\right)^H+\left(\nabla_{F^{\upsilon-1}X}Y\right)^v-\left(F^{\upsilon-1}\left(\nabla_X Y\right)\right)^v\} \end{aligned}$$

□

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References

[1] Alekseevsky, D., Grabowski, J., Marmo, G„ Michor, P.W., Poısson structures on the cotangent bundle of a Lie group or a prıncıple bundle and their reductions, *J. Math. Physics*, 35 (1994), 4909-4928.

[2] Akyol, M.A., Conformal anti-invariant submersions from cosymplectic manifolds, *Hacet. J. Math. Stat.* 46(2) (2017), 177-192.

[3] Akyol, M.A., Şahin, B., Conformal anti-invariant submersions from almost Hermitian manifolds, *Turkish Journal of Mathematics*, 40 (2016), 43-70.

[4] Brandt, H.E., Lorentz-invariant quantum fields in the space-time tangent bundle, *International Journal of Mathematics and Mathematical Sciences*, 24 (2003), 1529-1546.

[5] Cakan, R., On gh-lifts of some tensor fields, *Comptes rendus de l'Acade'mie bulgare des Sciences*, 71(3) (2018), 317-324.

[6] Çayır, H., Some notes on lifts of almost paracontact structures, *American Review of Mathematics and Statistics*, 3(1) (2015), 52-60.

[7] Çayır, H., Lie derivatives of almost contact structure and almost paracontact structure with respect to X^v and X^H on tangent bundle $T(M)$, *Proceedings of the Institute of Mathematics and Mechanics*, 42(1) (2016), 38-49.

[8] Çayır, H., Tachibana and Vishnevskii operators applied to X^v and X^H in almost paracontact structure on tangent bundle $T(M)$, *New Trends in Mathematical Sciences*, 4(3) (2016), 105-115.

[9] Çayır, H., Köseoğlu, G., Lie derivatives of almost contact structure and almost paracontact structure with respect to X^C and X^v on tangent bundle $T(M)$, *New Trends in Mathematical Sciences*, 4(1) (2016),153-159.

[10] Das, L.S., Prolongations of -structure to the tangent bundle of order, *International Journal of Mathematics and Mathematical Sciences*, 1(16) (1993), 201-204.

[11] Dube, K.K., On a differentiable structure satisfying $f^{2v+4}+f^2=0, f\neq 0$ and of type $(1,1)$, *The Nepali Math. Sc. Report*, 17(1 & 2) (1998), 99-102.

[12] Hou,Z.H., Sun,L., Slant curves in the unit tangent bundles of surfaces, *ISRN Geometry*, (2013) Article ID 821429.

[13] Kasap, Z., Weyl-Euler-Lagrange equations of motion on flat manifold, *Advances in Mathematical Physics*, (2015) Article ID 808016.

[14] Kim, J.B., Notes on $f-$manifold,*Tensor N-S*, 29 (1975), 299-302.

[15] Kobayashi, S., Nomizu, K., Foundations of Differential Geometry-Volume I., John Wiley & Sons, Inc, New York, 1963.

[16] Li, T., Krupka, D., The Geometry of Tangent Bundles: Canonical Vector Fields, *Geometry*, (2013) Article ID 364301.

[17] Das Lovejoy, S., Nivas, R., Pathak, V.N., On horizontal and complete lifts from a manifold with $f\lambda(7,1)-$structure to its cotangent bundle, *International Journal of Mathematics and Mathematical Sciences*, 8 (2005), 1291-1297.

[18] Nivas, R., Saxena, M., On complete and horizontal lifts from a manifold with HSU$-(4;2)$ structure to its cotangent bundle, *The Nepali Math. Sc. Report*, 23 (2004), 35-41.

[19] Salimov, A.A., Tensor Operators and Their applications, Nova Science Publ., New York, 2013.

[20] Sasaki, S., On the diferential geometry of tangent bundles of Riemannian manifolds. *Tohoku Math J.*, 10 (1958), 338-358.

[21] Srivastava, S.K., On the complete lifts of $(1,1)$ tensor field F satisfying structure $F^{v+1}-\lambda^2F^{v-1}=0$, *The Nepali Math. Sc. Report,* 21(1-2) (2003), 89-99.

[22] Salimov, A.A., Çayır, H., Some notes on almost paracontact structures, *Comptes Rendus de 1'Acedemie Bulgare Des Sciences*, 66(3) (2013), 331-338.

[23] Sahin, B., Semi-invariant Riemannian submersions from almost Hermitian manifolds, *Canad.Math. Bull.,* 56 (2013), 173-183.

[24] Upadhyay, M.D., Gupta, V.C., Integrability conditions of a $F(K, -(K-2))-$structure satisfying $F^K - F^{K-2} = 0$, $(F \neq 0, I)$, *Rev. Univ. Nac. Tucuman,* 20(1-2) (1976), 31-44.

[25] Yano,K., Ishihara, S., Tangent and Cotangent Bundles, Marcel Dekker Inc., New York, 1973.