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f-BIMINIMAL SUBMANIFOLDS OF GENERALIZED SPACE FORMS

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Abstract. We study f-biminimal submanifolds in generalized complex space forms and generalized Sasakian space forms. Then, we analyze f-biminimal submanifolds in these spaces. Finally, we consider f-biminimal integral submanifolds in Sasakian space forms and give an example.

1. Introduction

Harmonic map is a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds which is a critical point of the *energy functional*

$$E(\varphi) = \frac{1}{2} \int_{\Omega} \|d\varphi\|^2 \, d\nu_g,$$

where Ω is a compact domain of M. The Euler-Lagrange equation of energy functional $E(\varphi)$ is given by

$$\tau(\varphi) = tr(\nabla d\varphi) = 0,$$

where $\tau(\varphi)$ is the *tension field* of φ [4]. *A* map φ is called to be *biharmonic* if it is a critical point of the *bienergy functional*

$$E_2(\varphi) = \frac{1}{2} \int_{\Omega} \|\tau(\varphi)\|^2 \, d\nu_g,$$

where Ω is a compact domain of M. In [8], the Euler-Lagrange equation of bienergy functional $E_2(\varphi)$ is given by

$$\tau_2(\varphi) = tr(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla})\tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) = 0, \tag{1.1}$$

where $\tau_2(\varphi)$ is the *bitension field* of φ and R^N is the curvature tensor of N.

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A map φ is said to be *f-harmonic* with a function $f : M \overset{C^\infty}{\to} \mathbb{R}$ if it is a critical point of *f-energy functional*

$$E_f(\varphi) = \frac{1}{2}\int_\Omega f \left\| d\varphi \right\|^2 d\nu_g,$$

where Ω is a compact domain of M. In [3] and [15], the Euler-Lagrange equation of the f-harmonic functional $E_f(\varphi)$ is given by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(gradf) = 0, \tag{1.2}$$

where $\tau_f(\varphi)$ is the *f-tension field* of φ. The map φ is called to be *f-biharmonic* [12] if it is a critical point of the *f-bienergy functional*

$$E_{2,f}(\varphi) = \frac{1}{2}\int_\Omega f \left\| \tau(\varphi) \right\|^2 d\nu_g,$$

where Ω is a compact domain of M. The Euler-Lagrange equation of f-bienergy functional $E_{2,f}(\varphi)$ is given by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \Delta f\tau(\varphi) + 2\nabla^\varphi_{\mathrm{grad}\, f}\tau(\varphi) = 0, \tag{1.3}$$

where $\tau_{2,f}(\varphi)$ is called the *f-bitension field* of φ [12]. If f is a constant, an f-biharmonic map turns into a biharmonic map.

An immersion φ is called *biminimal* [11] if it is a critical point of the bienergy functional $E_2(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that φ is a critical point of the λ-bienergy

$$E_{2,\lambda}(\varphi) = E_2(\varphi) + \lambda E(\varphi) \tag{1.4}$$

for any smooth variation of the map $\varphi_t :]-\epsilon, +\epsilon[$, $\varphi_0 = \varphi$, such that $V = \frac{d\varphi_t}{dt}\mid_{t=0} = 0$ is normal to $\varphi(M)$. The Euler-Lagrange equation of λ-bienergy functional $E_{2,\lambda}(\varphi)$ is given by

$$[\tau_{2,\lambda}(\varphi)]^\perp = [\tau_2(\varphi)]^\perp - \lambda[\tau(\varphi)]^\perp = 0 \tag{1.5}$$

for some value of $\lambda \in \mathbb{R}$.

An immersion φ is called to be *f-biminimal* [7] if it is critical points of the f-bienergy functional $E_{2,f}(\varphi)$ and f-energy functional $E_f(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that φ is a critical point of the *λ-f-bienergy functional*

$$E_{2,\lambda,f}(\varphi) = E_{2,f}(\varphi) + \lambda E_f(\varphi)$$

for any smooth variation of the map $\varphi_t :]-\epsilon, +\epsilon[$, $\varphi_0 = \varphi$, such that $V = \frac{d\varphi_t}{dt}\mid_{t=0} = 0$ is normal to $\varphi(M)$. The Euler-Lagrange equation of λ-f-bienergy functional $E_{2,\lambda,f}(\varphi)$ is given by

$$[\tau_{2,\lambda,f}(\varphi)]^\perp = [\tau_{2,f}(\varphi)]^\perp - \lambda[\tau_f(\varphi)]^\perp = 0 \tag{1.6}$$

for some value of $\lambda \in \mathbb{R}$. It is called an immersion *free f-biminimal* if it is f-biminimal for $\lambda = 0$. If f is a constant, then the immersion is biminimal [7].

In [11], Loubeau and Montaldo defined biminimal immersions. They studied biminimal curves in a Riemannian manifold, curves in a space form, and isometric immersions of codimension 1 in a Riemannian manifold. In [7], the author and Özgür introduced f-biminimal immersions. They studied f-biminimal curves and hypersurfaces in a Riemannian manifold. In [17], Roth and Upadhyay studied biharmonic submanifolds in generalized space forms. In [18], the same authors studied necessary and sufficient conditions for f-biharmonicity and bi-f-harmonicity in generalized space forms. Motivated by the above studies, in the present paper, we consider f-biminimal submanifolds in generalized space forms. We find the necessary and sufficient conditions for submanifolds in generalized space forms to be f-biminimal.

2. Preliminaries

2.1. **Generalized complex space forms.** Let (N^{2n}, g, J) be an almost Hermitian manifold. The manifold (N^{2n}, g, J) is called *generalized complex space form* if its curvature tensor R is given by

$$\begin{aligned} R\,(X,Y)\,Z &= \alpha\,[g(Y,Z)X - g(X,Z)Y] \\ &+ \beta\,[g(JY,Z)JX - g(JX,Z)JY + 2g(JY,X)JZ]\,. \end{aligned} \tag{2.1}$$

where α and β are smooth functions on N [14],[19]. Assume that M be a submanifold of $N(\alpha,\beta)$ which is 4-dimensional generalized complex space form . Denote by J is an almost complex structure. It is easy to see that J satisfies

$$J^2 = -I \tag{2.2}$$

and

$$g(JX,Y) = -g(X,JY) \tag{2.3}$$

for X, Y tangent to $N(\alpha,\beta)$. Then we have

$$\nabla J = 0 \tag{2.4}$$

where ∇ means covariant derivation according to the Levi-civita connection.

Let $X \in TM$ and $\xi \in T^{\perp}M$. The decompositions of JX and $J\xi$ into tangent and normal components can be written as

$$JX = kX + hX \text{ and } J\xi = s\xi + t\xi, \tag{2.5}$$

where $k : TM \longrightarrow TM$, $h : TM \longrightarrow T^{\perp}M$, $s : T^{\perp}M \longrightarrow TM$, and $t : T^{\perp}M \longrightarrow T^{\perp}M$ are $(1,1)$-tensor fields. From equations (2.2) and (2.3), it is easy to see that k and t are skew-symmetric and satisfy the following properties:

$$k^2X = -X - shX, \tag{2.6}$$

$$t^2\xi = -\xi - hs\xi, \tag{2.7}$$

$$ks\xi + st\xi = 0, \tag{2.8}$$

$$hkX + thX = 0, \tag{2.9}$$

$$g(hX,\xi) = -g(X,s\xi) \tag{2.10}$$

for all $X \in TM$ and all $\xi \in T^{\perp}M$ [17].

2.2. **Generalized Sasakian space forms.** Let $\widetilde{M}^{2n+1} = \widetilde{M}(\varphi,\xi,\eta,\widetilde{g})$ be an almost contact metric manifold with almost contact metric structure $(\varphi,\xi,\eta,\widetilde{g})$. The notion of a generalized Sasakian space form is introduced by Alegre, Blair and Carriazo in [1]. The manifold $\widetilde{M}^{2n+1} = \widetilde{M}(\varphi,\xi,\eta,\widetilde{g})$ is called a *generalized Sasakian space form* if its curvature tensor $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z &= f_1\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}\\ &+f_2\{\widetilde{g}(X,\varphi Z)\varphi Y - \widetilde{g}(Y,\varphi Z)\varphi X + 2\widetilde{g}(X,\varphi Y)\varphi Z\}\\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \widetilde{g}(X,Z)\eta(Y)\xi - \widetilde{g}(Y,Z)\eta(X)\xi\}\end{aligned} \tag{2.11}$$

for certain differentiable functions f_1, f_2 and f_3 on $\widetilde{M}^{2n+1}$ [1]. The typical examples of generalized Sasakian space forms with constant functions are a Sasakian space form $\left(f_1 = \frac{c+3}{4},\ f_2 = f_3 = \frac{c-1}{4}\right)$ [2], a Kenmotsu space form $\left(f_1 = \frac{c-3}{4},\ f_2 = f_3 = \frac{c+1}{4}\right)$ [9], a cosymplectic space form $\left(f_1 = f_2 = f_3 = \frac{c}{4}\right)$ [13].

Let (M,g) be a submanifold of an almost contact metric manifold $\widetilde{M}^{2n+1}$. Let $X \in TM$ and $\vartheta \in T^{\perp}M$. The decompositions of φX and $\varphi\vartheta$ into tangent and normal components can be written as

$$\varphi X = PX + NX \text{ and } \varphi\vartheta = t\vartheta + s\vartheta, \tag{2.12}$$

where $P : TM \longrightarrow TM$, $N : TM \longrightarrow T^{\perp}M$, $t : T^{\perp}M \longrightarrow TM$, and $s : T^{\perp}M \longrightarrow T^{\perp}M$ are $(1,1)$-tensor fields. A submanifold M of a generalized Sasakian space form $\widetilde{M}^{2n+1}$ is called *anti-invariant* (resp. *invariant*) if P (resp.N) vanishes identically. Moreover, it is known that $\varphi(T_XM) \subset T_X^{\perp}M$ for all $X \in M$, then M is anti-invariant [10], [20]. A submanifold M of a Sasakian space form N^{2n+1} is called an *integral submanifold* if $\eta(X) = 0$ for any vector field X tangent to M [2].

3. f-Biminimal submanifolds of generalized complex space forms

Let $N(\alpha,\beta)$ be a generalized complex space form and M^n an $n < 4$-dimensional submanifold of $N(\alpha,\beta)$ and denote by B, A, H, $\nabla^{\perp}$ and $\Delta^{\perp}$, the second fundamental form, the shape operator, the mean curvature vector field, the connection and the Laplacian in normal bundle, respectively.

We have the following theorem:

Theorem 3.1. *Let M^n be a submanifold of a generalized complex space form $N(\alpha,\beta)$. The submanifold $i : M^n \to N(\alpha,\beta)$ is f-biminimal if and only if*

$$-\Delta^{\perp}H + traceB(.,A_H.) - n\alpha H + 3\beta hsH - \lambda H + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\operatorname{grad}\ln f}H = 0. \tag{3.1}$$

Proof. Let $\{e_i\}$, $1 \leq i \leq n$ be a local geodesic orthonormal frame at $p \in M$. From [5], [6] and [16], it is clear that the normal parts of the tension field, the bitension field and f-bitension field of i are

$$[\tau(i)]^{\perp} = nH, \tag{3.2}$$

$$[\tau_2(i)]^{\perp} = n\left\{-\Delta^{\perp}H + traceB(.,A_H.) + \left(\sum_{i=1}^{n} R^N(e_i,H)e_i\right)^{\perp}\right\} \tag{3.3}$$

and

$$[\tau_{2,f}(\varphi)]^{\perp} = f\,[\tau_2(\varphi)]^{\perp} + \Delta f\,[\tau(\varphi)]^{\perp} + 2\left[\nabla^{\varphi}_{\mathrm{grad}\,f}\tau(\varphi)\right]^{\perp}. \tag{3.4}$$

Using the equation (3.2) into (1.2), we can write

$$[\tau_f(\varphi)]^{\perp} = f\,[\tau(\varphi)]^{\perp} = fnH. \tag{3.5}$$

From the equation (2.1), after a straightforward computation, we have

$$\left(\sum_{i=1}^{n} R^N(e_i,H)e_i\right)^{\perp} = -\alpha nH + 3\beta hsH. \tag{3.6}$$

Then, putting the equation (3.6) into equation (3.3), we can write

$$[\tau_2(i)]^{\perp} = n\left\{-\Delta^{\perp}H + traceB(.,A_H.) - \alpha nH + 3\beta hsH\right\}. \tag{3.7}$$

From the Weingarten formula, we have

$$\left[\nabla^{\varphi}_{\mathrm{grad}\,f}\tau(\varphi)\right]^{\perp} = \left[\nabla^{\varphi}_{\mathrm{grad}\,f}nH\right]^{\perp} = n\nabla^{\perp}_{\mathrm{grad}\,f}H. \tag{3.8}$$

Putting the equations (3.2), (3.7) and (3.8) into (3.4), we find

$$\begin{aligned}[\tau_{2,f}(\varphi)]^{\perp} &= nf\left(-\Delta^{\perp}H + traceB(.,A_H.) - \alpha nH + 3\beta hsH\right)\\ &\quad + nf\,(\Delta f)\,H + 2n\nabla^{\perp}_{\mathrm{grad}\,f}H.\end{aligned} \tag{3.9}$$

Finally, substituting the equations (3.5) and (3.9) into the equation (1.6), we obtain

$$nf\left\{-\Delta^{\perp}H + traceB(.,A_H.) - n\alpha H + 3\beta hsH - \lambda H + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H\right\} = 0.$$

This completes the proof. □

Corollary 3.1. *Let M^n be a submanifold with $n < 4$ of a generalized complex space form $N(\alpha,\beta)$.*

1) *M^n is an f-biminimal hypersurface if and only if*

$$-\Delta^{\perp}H + traceB(.,A_H.) - \left(3\alpha + 3\beta + \lambda - \frac{\Delta f}{f}\right)H + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = 0.$$

2) *M^n is an f-biminimal complex surface if and only if*

$$-\Delta^{\perp}H + traceB(.,A_H.) - \left(2\alpha + \lambda - \frac{\Delta f}{f}\right)H + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = 0.$$

3) *M^n is an f-biminimal Lagrangian surface if and only if*

$$-\Delta^{\perp}H + traceB(.,A_H.) - \left(2\alpha + 3\beta + \lambda - \frac{\Delta f}{f}\right)H + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = 0.$$

4) *M^n is an f-biminimal curve if and only if*

$$-\Delta^{\perp}H + traceB(.,A_H.) - \left(\alpha + 3\beta + 3\beta t^2 + \lambda - \frac{\Delta f}{f}\right)H + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = 0.$$

Proof. 1) Since M^n is a hypersurface, we have $t = 0$ and $n = 3$. By the use of equation (2.7), we have $ksH = -H$. From Theorem 3.1, we get the result.

2) Since M^n is a complex surface, we get $k = 0$, $s = 0$ and $n = 2$. Using Theorem 3.1, we obtain the result.

3) Since M^n is a Lagrangian surface, we have $k = 0$, $t = 0$ and $n = 2$. By the use of equation (2.7), we have $hsH = -H$. From Theorem 3.1, we get the result.

4) Since M^n is a curve, we get $k = 0$ and $n = 1$. By the use of equation (2.7), we have $hsH = -\left(H + t^2H\right)$. Using Theorem 3.1, we obtain the result.

This completes the proof. □

As an immediate consequence of the above corollary for curves and complex or Lagrangian surfaces with parallel mean curvature, we have:

Corollary 3.2. *Let M^n be a submanifold with $n < 4$ of a generalized complex space form $N(\alpha,\beta)$.*

1) *M^n is an f-biminimal complex surface with parallel mean curvature if and only if*

$$traceB(.,A_H.) = \left(2\alpha + \lambda - \frac{\Delta f}{f}\right)H.$$

2) *M^n is an f-biminimal Lagrangian surface with parallel mean curvature if and only if*

$$traceB(.,A_H.) = \left(2\alpha + 3\beta + \lambda - \frac{\Delta f}{f}\right)H.$$

3) *M^n is an f-biminimal curve with parallel mean curvature if and only if*

$$traceB(.,A_H.) = \left(\alpha + \lambda - \frac{\Delta f}{f}\right)H + 3\beta\left(H + t^2H\right).$$

Now, we have the following proposition for hypersurfaces with constant mean curvature in a generalized complex space form $N(\alpha,\beta)$.

Proposition 3.1. *Let M^3 be a hypersurface of a generalized complex space form $N(\alpha,\beta)$ with non-zero constant mean curvature H. Then M^3 is f-biminimal if and only if*

$$\|B\|^2 = 3\alpha + 3\beta + \lambda - \frac{\Delta f}{f}$$

and the scalar curvature of M^3 satisfies

$$Scal_M = 3\alpha + 3\beta - \lambda + \frac{\Delta f}{f} + 9H^2.$$

Proof. Assume that M^3 is a hypersurface, from Corollary 3.1, M^3 is f-biminimal if and only if

$$-\Delta^{\perp} H + traceB(., A_H.) - \left(3\alpha + 3\beta + \lambda - \frac{\Delta f}{f}\right) H + 2\nabla^{\perp}_{\operatorname{grad}\ln f} H = 0.$$

Since M^3 has constant mean curvature, we can write

$$traceB(., A_H.) = \left(3\alpha + 3\beta + \lambda - \frac{\Delta f}{f}\right) H.$$

In addition, for hypersurfaces, it is clear that $A_H = HA$. Then, we get

$$H \|B\|^2 = \left(3\alpha + 3\beta + \lambda - \frac{\Delta f}{f}\right) H.$$

Since H is a non-zero constant mean curvature, we get

$$\|B\|^2 = \left(3\alpha + 3\beta + \lambda - \frac{\Delta f}{f}\right) \tag{3.10}$$

By the use of the Gauss equation, we obtain

$$Scal_M = \sum_{i,j=1}^{3} g\left(R^N(e_i, e_j)e_j, e_i\right) + 9H^2 - \|B\|^2 \tag{3.11}$$

where $\{e_i\}$, $1 \leq i \leq 3$ be a local geodesic orthonormal frame at $p \in M$. Using equation (2.1), we can write

$$\sum_{i,j=1}^{3} g\left(R^N(e_i, e_j)e_j, e_i\right) = \alpha \sum_{i,j=1}^{3} \left[g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)^2\right]$$

$$+\beta \sum_{i,j=1}^{3} \left[g(Je_j, e_j)g(Je_i, e_i) - g(Je_i, e_j)g(Je_j, e_i) + 2g(Je_j, e_i)g(Je_j, e_i)\right].$$

Hence, we find

$$\sum_{i,j=1}^{3} g\left(R^N(e_i, e_j)e_j, e_i\right) = 6\alpha + 6\beta. \tag{3.12}$$

Finally, in view of equations (3.10) and (3.12) into (3.11), we get

$$Scal_M = 3\alpha + 3\beta - \lambda + \frac{\Delta f}{f} + 9H^2.$$

This proves the proposition. □

For Lagrangian surfaces of $N(\alpha, \beta)$, we can state the following proposition:

Proposition 3.2. *Let M^2 be a Lagrangian surface of $N(\alpha,\beta)$ with non-zero constant mean curvature H.*

1) *If M^2 is f-biminimal, then*

$$0 < \|H\|^2 \leq \inf\left(\frac{2\alpha+3\beta+\lambda-\frac{\Delta f}{f}}{2}\right). \tag{3.13}$$

2) *Assume that f is an eigenfunction of the Laplacian Δ corresponding to real eigenvalue μ. Hence the equality in (3.13) occurs and M^2 is f-biminimal if and only if M^2 is pseudo-umbilical and $\nabla^\perp H = 0$.*

Proof. Let M^2 be a Lagrangian surface. From Corollary (3.1), we have

$$-\Delta^\perp H + traceB(.,A_H.) - \left(2\alpha+3\beta+\lambda-\frac{\Delta f}{f}\right)H + 2\nabla^\perp_{\operatorname{grad}\ln f}H = 0. \tag{3.14}$$

Then taking the scalar product of equation (3.14) with H, we find

$$-g(\Delta^\perp H,H)+g(trB(.,A_H(.)),H)-\left(2\alpha+3\beta+\lambda-\frac{\Delta f}{f}\right)g(H,H)+2g\left(\nabla^\perp_{\operatorname{grad}\ln f}H,H\right)=0.$$

Since $\|H\|$ is a constant, we have

$$-g(\Delta^\perp H,H) + \|A_H\|^2 = \left(2\alpha+3\beta+\lambda-\frac{\Delta f}{f}\right)\|H\|^2.$$

Using the Bochner formula, we get

$$\left\|\nabla^\perp H\right\|^2 + \|A_H\|^2 = \left(2\alpha+3\beta+\lambda-\frac{\Delta f}{f}\right)\|H\|^2. \tag{3.15}$$

By the use of Cauchy-Schwarz inequality, we have $\|A_H\|^2 \geq 2\|H\|^4$. Hence, we find

$$\left(2\alpha+3\beta+\lambda-\frac{\Delta f}{f}\right)\|H\|^2 \geq 2\|H\|^4 + \left\|\nabla^\perp H\right\|^2 \geq 2\|H\|^4. \tag{3.16}$$

Since $\|H\|$ is a non-zero constant, we can write

$$0 < \|H\|^2 \leq \inf\left(\frac{2\alpha+3\beta+\lambda-\frac{\Delta f}{f}}{2}\right). \tag{3.17}$$

Now, if f is an eigenfunction of the Laplacian Δ corresponding to the real eigenvalue μ, then $\frac{\Delta f}{f} = \mu$. We can write

$$\|H\|^2 = \left(\frac{2\alpha+3\beta+\lambda-\mu}{2}\right). \tag{3.18}$$

Assume that M^2 is f-biminimal. From (3.16), we obtain $\nabla^\perp H = 0$. In addition, substituting the equation (3.18) into (3.16), we get

$$\|A_H\|^2 = \frac{(2\alpha+3\beta+\lambda-\mu)^2}{2}.$$

That is, M^2 is pseudo-umbilical. This completes the proof. □

Remark 1. *Let M^2 be a Lagrangian surface of a generalized complex space form $N(\alpha,\beta)$ with non-zero constant mean curvature H.*

Remark 3.1. 1) *If* $\inf\left(2\alpha+3\beta+\lambda-\frac{\Delta f}{f}\right)$ *is non-positive then M^2 is not f-biminimal.*

2) *Using the Proposition 3.8 in* [18], *we obtain that if* $\inf\left(2\alpha+3\beta-\frac{\Delta f}{f}\right)$ *is non-positive and* $\lambda>\left|2\alpha+3\beta-\frac{\Delta f}{f}\right|$ *then M^2 is f-biminimal and not f-biharmonic.*

For complex surfaces of $N(\alpha,\beta)$, we can state the following proposition:

Proposition 3.3. *Let M^2 be a complex surface of the generalized complex space form $N(\alpha,\beta)$ with non-zero constant mean curvature H.*

1) *If M^2 is f-biminimal, then*

$$0<\|H\|^2\le\inf\left(\frac{2\alpha+\lambda-\frac{\Delta f}{f}}{2}\right). \tag{3.19}$$

2) *Assume that f is an eigenfunction of the Laplacian Δ corresponding to real eigenvalue μ. Hence the equality in (3.19) occurs and M^2 is f-biminimal if and only if M^2 is pseudo-umbilical and $\nabla^{\perp}H=0$.*

Proof. By the same method in the proof of Proposition (3.2), we get the result. □

Remark 3.2. *Let M^2 be a complex surface of the generalized complex space form $N(\alpha,\beta)$ with non-zero constant mean curvature H.*

1) *If* $\inf\left(2\alpha+\lambda-\frac{\Delta f}{f}\right)$ *is non-positive then M^2 is not f-biminimal.*

2) *Using the Proposition 3.9 in* [18], *we obtain that if* $\inf\left(2\alpha-\frac{\Delta f}{f}\right)$ *is non-positive and* $\lambda>\left|2\alpha-\frac{\Delta f}{f}\right|$ *then M^2 is f-biminimal and not f-biharmonic.*

4. f-Biminimal submanifolds of generalized Sasakian space forms

Let $\widetilde{M}^{2n+1}=\widetilde{M}(\varphi,\xi,\eta,\widetilde{g})$ be a generalized Sasakian space form and (M^n,g) an n-dimensional submanifold of $\widetilde{M}^{2n+1}$ and denote by B, A, H, $\nabla^{\perp}$ and $\Delta^{\perp}$, the second fundamental form, the shape operator, the mean curvature vector field, the connection and the Laplacian in normal bundle, respectively.

We have the following theorem:

Theorem 4.1. *Let M^n be a submanifold of a generalized Sasakian space form $\widetilde{M}^{2n+1}$. The submanifold $i: M^n\to\widetilde{M}^{2n+1}$ is f-biminimal if and only if*

$$-\Delta^{\perp}H+traceB(.,A_H.)-\left(nf_1+\lambda-\frac{\Delta f}{f}\right)H+3f_2NtH+f_3\left|\xi^{\intercal}\right|^2H$$

$$+ nf_3\eta(H)\xi^{\perp} + 2\nabla^{\perp}_{\operatorname{grad}\ln f}H = 0. \tag{4.1}$$

Proof. Let $\{e_i\}$, $1 \leq i \leq n$ be a local geodesic orthonormal frame at $p \in M$. From the equation (2.11), after a straightforward computation, we have

$$\begin{aligned}\widetilde{R}(e_i, H)e_i &= -f_1\widetilde{g}(e_i, e_i)H - 3f_2\widetilde{g}(H, \varphi e_i)\varphi e_i \\ &+ f_3\left[\eta(e_i)^2 H - \eta(H)\eta(e_i)e_i + \widetilde{g}(e_i, e_i)\eta(H)\xi\right]. \end{aligned}\tag{4.2}$$

Using the equation (2.12), we obtain

$$\begin{aligned}\sum_{i=1}^{n}\widetilde{R}(e_i, H)e_i &= -nf_1H + 3f_2\left[PtH + NtH\right] \\ &+ f_3\left[|\xi^{\intercal}|^2 H - \eta(H)\xi^{\intercal} + n\eta(H)\xi\right]. \end{aligned}\tag{4.3}$$

Hence, we have

$$\left(\sum_{i=1}^{n}\widetilde{R}(e_i, H)e_i\right)^{\perp} = -nf_1H + 3f_2(NtH) + f_3\left[|\xi^{\intercal}|^2 H + n\eta(H)\xi^{\perp}\right]. \tag{4.4}$$

Then, putting the equation (4.4) into equation (3.3), we can write

$$\begin{aligned}[\tau_2(i)]^{\perp} &= n\left\{-\Delta^{\perp}H + traceB(., A_H.) - nf_1H + 3f_2(NtH)\right. \\ &\left.+ f_3\left[|\xi^{\intercal}|^2 H + n\eta(H)\xi^{\perp}\right]\right\}. \end{aligned}\tag{4.5}$$

Putting the equations (3.2), (3.8) and (4.5) into (3.4), we find

$$\begin{aligned}[\tau_{2,f}(\varphi)]^{\perp} &= nf\left(-\Delta^{\perp}H + traceB(., A_H.) - nf_1H + 3f_2(NtH)\right) \\ &+ (nf)f_3\left[|\xi^{\intercal}|^2 H + n\eta(H)\xi^{\perp}\right] + nf(\Delta f)H + 2n\nabla^{\perp}_{\operatorname{grad} f}H. \end{aligned}\tag{4.6}$$

Finally, substituting equations (3.5) and (4.6) into equation (1.6), we obtain

$$\begin{aligned} nf&\left\{-\Delta^{\perp}H + traceB(., A_H.) - \left(nf_1 + \lambda - \frac{\Delta f}{f}\right)H + 3f_2(NtH)\right. \\ &\left.+ f_3\left[|\xi^{\intercal}|^2 H + n\eta(H)\xi^{\perp}\right] + 2\nabla^{\perp}_{\operatorname{grad}\ln f}H\right\} = 0. \end{aligned}$$

This completes the proof. □

Corollary 4.1. *Let M^n be a submanifold of a generalized Sasakian space form $\widetilde{M}^{2n+1}$.*

1) *If M^n is invariant, then M^n is f-biminimal if and only if*

$$\begin{aligned}-\Delta^{\perp}H + traceB(., A_H.) + 2\nabla^{\perp}_{\operatorname{grad}\ln f}H &= \left(nf_1 + \lambda - \frac{\Delta f}{f}\right)H \\ &-f_3|\xi^{\intercal}|^2 H - nf_3\eta(H)\xi^{\perp}.\end{aligned}$$

2) *If ξ is normal to M^n, then M^n is f-biminimal if and only if*

$$-\Delta^{\perp}H + traceB(.,A_H.) + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = \left(nf_1 + \lambda - \frac{\Delta f}{f}\right)H$$
$$-3f_2NtH - nf_3\eta(H)\xi.$$

3) *If ξ is tangent to M^n, then M^n is f-biminimal if and only if*

$$-\Delta^{\perp}H + traceB(.,A_H.) + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = \left(nf_1 - f_3 + \lambda - \frac{\Delta f}{f}\right)H - 3f_2NtH.$$

4) *If M^{2n} is a hypersurface, then M^{2n} is f-biminimal if and only if*

$$-\Delta^{\perp}H + traceB(.,A_H.) + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = \left(2nf_1 + 3f_2 + \lambda - \frac{\Delta f}{f}\right)H$$
$$-(3f_2 + 2nf_3)\eta(H)\xi^{\perp} - f_3|\xi^{\intercal}|^2 H.$$

Proof. 1) Assume that M^n is invariant, then we have $N = 0$. From Theorem 4.1, we obtain the result.

2) If ξ is normal to M^n, then M^n is anti-invariant, $\xi^{\perp} = \xi$ and $\xi^{\intercal} = 0$. From Theorem 4.1, we obtain this case.

3) If ξ is tangent to M^n, then $\xi^{\perp} = 0$ and $\xi^{\intercal} = \xi$ and $|\xi| = 1$. From Theorem 4.1, we find this case.

4) Assume that M^{2n} is a hypersurface. Hence, we have $\varphi(H)$ is tangent and $sH = 0$. Then, we obtain $-H + \eta(H)\xi = PtH + NtH$. Hence comparing the tangential and normal parts, $NtH = -H + \eta(H)\xi^{\perp}$ and $PtH = \eta(H)\xi^{\intercal}$ which gives the result. □

Proposition 4.1. *Let M^{2n} be a hypersurface of a generalized Sasakian space form $\widetilde{M}^{2n+1}$ with non-zero constant mean curvature H such that ξ is tangent to M^{2n}. Then M^{2n} is f-biminimal if and only if*

$$\|B\|^2 = 2nf_1 + 3f_2 - f_3 + \lambda - \frac{\Delta f}{f}$$

and the scalar curvature of M^{2n} satisfies

$$Scal_M = 2n(2n-2)f_1 + 6(n-1)f_2 - (4n-3)f_3 - \lambda + 4n^2H^2 + \frac{\Delta f}{f}.$$

Proof. Suppose that M^{2n} is a hypersurface, from Corollary 4.1, M^{2n} is f-biminimal if and only if

$$-\Delta^{\perp}H + traceB(.,A_H.) + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = \left(2nf_1 + 3f_2 + \lambda - \frac{\Delta f}{f}\right)H$$
$$-(3f_2 + 2nf_3)\eta(H)\xi^{\perp} - f_3|\xi^{\intercal}|^2 H.$$

Since M^{2n} has constant mean curvature, we can write

$$trace B(.,A_H.) = \left(2nf_1 + 3f_2 + \lambda - \frac{\Delta f}{f}\right) H - (3f_2 + 2nf_3)\, \eta(H)\, \xi^{\perp} - f_3 \left|\xi^{\intercal}\right|^2 H.$$

Using Lemma 4.4 in [17], we have $Pt = 0$ and $Nt = -I$. Suppose that ξ is tangent to M^{2n}, then it is known that $\xi^{\perp} = 0$, $\xi^{\intercal} = \xi$ and $|\xi| = 1$. Hence,

$$trace B(.,A_H.) = \left(2nf_1 + 3f_2 - f_3 + \lambda - \frac{\Delta f}{f}\right) H.$$

In addition, H is a non-zero constant and it is clear that $A_H = HA$ for hypersurfaces. Then, we get

$$\|B\|^2 = \left(2nf_1 + 3f_2 - f_3 + \lambda - \frac{\Delta f}{f}\right). \tag{4.7}$$

Using the Gauss equation, we obtain

$$Scal_M = \sum_{i,j=1}^{2n} g\left(\widetilde{R}(e_i, e_j)e_j, e_i\right) + (2n)^2 H^2 - \|B\|^2 \tag{4.8}$$

where $\{e_i\}$, $1 \le i \le 2n$ be a local geodesic orthonormal frame at $p \in M$. By the use of equation (2.11), we obtain

$$\sum_{i,j=1}^{2n} \widetilde{g}\left(R^N(e_i, e_j)e_j, e_i\right) = f_1 \sum_{i,j=1}^{2n} \left[\widetilde{g}(e_j, e_j)\widetilde{g}(e_i, e_i) - \widetilde{g}(e_i, e_j)^2\right]$$

$$+ f_2 \sum_{i,j=1}^{2n} \left[\widetilde{g}(e_i, \varphi e_j)\widetilde{g}(\varphi e_j, e_i) - \widetilde{g}(e_j, \varphi e_j)\widetilde{g}(\varphi e_i, e_i) + 2\widetilde{g}(e_i, \varphi e_j)\widetilde{g}(\varphi e_j, e_i)\right]$$

$$+ f_3 \sum_{i,j=1}^{2n} \left[\eta(e_i)\, \eta(e_j)\, \widetilde{g}(e_j, e_i) - \eta(e_j)\, \eta(e_j)\, \widetilde{g}(e_i, e_i)\right.$$

$$\left. + \widetilde{g}(e_i, e_j)\eta(e_i)\, \eta(e_j) - \widetilde{g}(e_j, e_j)\eta(e_i)\, \eta(e_i)\right].$$

Hence, we find

$$\sum_{i,j=1}^{2n} g\left(R^N(e_i, e_j)e_j, e_i\right) = 2n\,(2n-1)\, f_1 + 3\,(2n-1)\, f_2 + f_3\,(2-4n). \tag{4.9}$$

Finally, in view of equations (4.7) and (4.9) into (4.8), we get

$$Scal_M = 2n\,(2n-2)\, f_1 + 6\,(n-1)\, f_2 - f_3\,(4n-3) - \lambda + 4n^2 H^2 + \frac{\Delta f}{f}.$$

This proves the proposition. □

Remark 4.1. *Let M^{2n} be a constant mean curvature hypersurface of generalized Sasakian space form $\widetilde{M}^{2n+1}$ with tangent ξ.*

1) *If the functions f_1, f_2, f_3 satisfy the inequality $2nf_1 + 3f_2 - f_3 + \lambda \leq \frac{\Delta f}{f}$ on M then M is not f-biminimal.*

2) *Using the Corollary 3.13 in* [18], *we obtain that if $2nf_1 + 3f_2 - f_3 - \frac{\Delta f}{f} \leq 0$ and $\lambda > \left|2nf_1 + 3f_2 - f_3 - \frac{\Delta f}{f}\right|$ then M is f-biminimal and not f-biharmonic.*

5. f-Biminimal integral submanifolds of Sasakian space forms

In the present section, we consider f-biminimal integral submanifolds in Sasakian space forms and give an example. Now, we have the following theorem:

Theorem 5.1. *Let M^n be a submanifold of a Sasakian space form N^{2n+1}. The integral submanifold $i : M^n \to N^{2n+1}$ is f-biminimal if and only if*

$$-\Delta^{\perp}H + traceB(., A_H.) - \left(nf_1 + \lambda - \frac{\Delta f}{f}\right)H + 3f_2H + 2\nabla^{\perp}_{\mathrm{grad}\ln f}H = 0.$$

Proof. Using the Theorem 4.1 and definition of integral submanifold, we obtain the desired result. □

To obtain an example of f-biminimal integral submanifolds, similar to the proof of Theorem 4.1, Remark 4.2 and Theorem 4.3 in [6], we state the following Theorem 5.2, Remark 5.1 and Theorem 5.3:

Theorem 5.2. *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian space form with constant φ-sectional curvature c and $i : M \to N$ an r-dimensional integral submanifold of N, $1 \leq r \leq n$. Consider*

$$F : \widetilde{M} = I \times M \to N \quad , \quad F(t,p) = \phi_t(p) = \phi_p(t),$$

where $I = \mathbb{S}^1$ or $I = \mathbb{R}$ and $\{\phi_t\}_{t\in I}$ is the flow of the vector field ξ. Then F : $\left(\widetilde{M}, \widetilde{g} = dt^2 + i^*g\right) \to N$ is a Riemannian immersion [6]*. Then $\widetilde{M}$ is f-biminimal if and only if M is a f-biminimal submanifold of N, where $f : M \to \mathbb{R}$ is a differentiable function.*

Proof. By [6], we have

$$\tau(F)_{(t,p)} = (d\phi_t)_p \, \tau(i) \tag{5.1}$$

and

$$\tau_2(F)_{(t,p)} = (d\phi_t)_p \, \tau_2(i). \tag{5.2}$$

Let $\sigma \in C(F^{-1}(TN))$ be a section in $F^{-1}(TN)$ defined by

$$\sigma_{(t,p)} = (d\phi_t)_p \, (Z_p), \tag{5.3}$$

where Z is a vector field along M. Then we have

$$\left(\nabla^F_X \sigma\right)_{(t,p)} = (d\phi_t)_p \left(\nabla^N_X Z\right) \quad , \quad \forall X \in C(TM), \tag{5.4}$$

where ∇^F is the pull-back connection determined by the Levi-Civita connection on N (see [6]). Using the equations (5.1) and (5.4), we calculate

$$\begin{aligned}\nabla^F_{gradf}\tau(F) &= \nabla^F_{gradf}\left((d\phi_t)_p\,\tau(i)\right)\\ &= (d\phi_t)_p\,\nabla^i_{gradf}\tau(i).\end{aligned}\tag{5.5}$$

In view of the equations (5.1), (5.2) and (5.5) into the equation (2.3), we get

$$\left[\tau_{2,f}(F)_{(t,p)}\right]^{\perp} = (d\phi_t)_p\,[\tau_{2,f}(i)]^{\perp}.\tag{5.6}$$

Using the equations (5.1) in (1.2), we obtain

$$\left[\tau_{f}(F)_{(t,p)}\right]^{\perp} = (d\phi_t)_p\,[\tau_{f}(i)]^{\perp}.\tag{5.7}$$

By the use of the equations (5.6), (5.7) in (1.6), we find

$$\begin{aligned}[\tau_{2,\lambda,f}(F)_{(t,p)}]^{\perp} &= \left[\tau_{2,f}(F)_{(t,p)}\right]^{\perp} - \lambda\left[\tau_{f}(F)_{(t,p)}\right]^{\perp}\\ &= (d\phi_t)_p\left\{[\tau_{2,f}(i)]^{\perp} - \lambda\,[\tau_f(i)]^{\perp}\right\} = (d\phi_t)_p\,[\tau_{2,\lambda,f}(i)]^{\perp}.\end{aligned}$$

This completes the proof. □

By the use of f-biminimality of F and Fubini Theorem, we have

Remark 5.1. *Let $(N^{2n+1},\varphi,\xi,\eta,g)$ be a compact strictly regular Sasakian manifold and $G: M \to N$ be an arbitrary smooth map from a compact Riemannian manifold M. If F is f-biminimal, then G is f-biminimal, where*

$$F:\widetilde{M} = \mathbb{S}^1 \times M \to N \quad , \quad F(t,p) = \phi_t(G(p)).$$

Using the above remark, we can state the following theorem:

Theorem 5.3. *Let $N^{2n+1}(c)$ be a Sasakian space form with constant φ-sectional curvature c and $\widetilde{M}^2$ a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then $\widetilde{M}$ is f-biminimal if and only if, locally, it is given by $F(t,s) = \phi_t(\gamma(s))$, where γ is a f-biminimal Legendre curve.*

In [7], it is given by an example of f-biminimal Legendre curve in $\mathbb{R}^5(-3)$:

Example 5.1. ([7]) *Let us take $\gamma(t) = (\sin 2t, -\cos 2t, 0, 0, 1)$ in $\mathbb{R}^5(-3)$. The curve γ is an f-biminimal Legendre curve with osculating order $r = 2$, $k_1 = 2$, $f = e^t$, $\varphi T \perp E_2$. The curve γ is not f-biharmonic. For $\lambda \neq -4$, it is easy to see that γ is not biminimal.*

Using Example 5.1 and Theorem 5.3, we can give the following example of f-biminimal surfaces:

Example 5.2. *Let $\widetilde{M}^2$ be a surface of $\mathbb{R}^5(-3)$ endowed with its canonical Sasakian structure which is invariant under the flow-action of the characteristic vector field ξ. If γ is a Legendre curve given in Example 5.1 and locally, $\widetilde{M}^2$ is given by $F(t,s) = \phi_t(\gamma(s))$, then $\widetilde{M}^2$ is f-biminimal. Since γ is not f-biharmonic, $\widetilde{M}^2$ is not f-biharmonic.*

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