



## ON ALMOST $\alpha$ -PARA-KENMOTSU MANIFOLDS SATISFYING CERTAIN CONDITIONS

I. KÜPELİ ERKEN

**ABSTRACT.** In this paper, we study some remarkable properties of almost  $\alpha$ -para-Kenmotsu manifolds. We consider projectively flat, conformally flat and concircularly flat almost  $\alpha$ -para-Kenmotsu manifolds (with the  $\eta$ -parallel tensor field  $\phi h$ ). Finally, we present an example to verify our results.

### 1. INTRODUCTION

The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [7] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in paper of Zamkovoy, [17]. However such manifolds were studied earlier, [1], [3], [4], [11]. These authors called such structures almost para-coHermitian. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [5], [6], [13], [17].

Considering the recent stage of the developments in the theory, there is an impression that the geometers are focused on problems in almost paracontact metric geometry which are created ad hoc.

Almost (para)contact metric structure is given by a pair  $(\eta, \Phi)$ , where  $\eta$  is a 1-form,  $\Phi$  is a 2-form and  $\eta \wedge \Phi^n$  is a volume element. It is well known that then there exists a unique vector field  $\xi$ , called the characteristic (Reeb) vector field, such that  $i_\xi \eta = 1$ ,  $i_\xi \Phi = 0$ . The Riemannian or pseudo-Riemannian geometry appears if we try to introduce a *compatible* structure which is a metric or pseudo-metric  $g$  and an affiner  $\phi$  ((1,1)-tensor field), such that

$$\Phi(X, Y) = g(\phi X, Y), \quad \phi^2 = \epsilon(Id - \eta \otimes \xi). \quad (1.1)$$

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We have almost paracontact metric structure for  $\epsilon = +1$  and almost contact metric for  $\epsilon = -1$ . Then, the triple  $(\phi, \xi, \eta)$  is called almost paracontact structure or almost contact structure, resp.

Combining the assumption concerning the forms  $\eta$  and  $\Phi$ , we obtain many different types of almost (para)contact manifolds, e.g. (para)contact if  $\eta$  is contact form and  $d\eta = \Phi$ , almost (para)cosymplectic if  $d\eta = 0$ ,  $d\Phi = 0$ , almost (para)Kenmotsu if  $d\eta = 0$ ,  $d\Phi = 2\eta \wedge \Phi$ .

Classifications are obtained for contact metric, almost cosymplectic, almost  $\alpha$ -Kenmotsu and almost  $\alpha$ -cosymplectic manifolds, e.g [9], [10], for paracontact case [8].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let  $M$  be a  $(2n + 1)$ -dimensional semi-Riemannian manifold with metric  $g$ . The Ricci operator  $Q$  of  $(M, g)$  is defined by  $g(QX, Y) = S(X, Y)$ , where  $S$  denotes the Ricci tensor of type  $(0, 2)$  on  $M$ . If there exists a one-to-one correspondence between each coordinate neighborhood of  $M$  and a domain in Euclidian space such that any geodesic of the semi-Riemannian manifold corresponds to a straight line in the Euclidean space, then  $M$  is said to be locally projectively flat. For  $n \geq 1$ ,  $M$  is locally projectively flat if and only if the well known projective curvature tensor  $P$  vanishes. Here  $P$  is defined by [12]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y] \quad (1.2)$$

for all  $X, Y, Z \in T(M)$ , where  $R$  is the curvature tensor and  $S$  is the Ricci tensor.

In fact  $M$  is projectively flat if and only if it is of constant curvature [16]. Thus the projective curvature tensor is the measure of the failure of a semi-Riemannian manifold to be of constant curvature.

In semi-Riemannian geometry, one of the basic interest is curvature properties and to what extend these determine the manifold itself. One of the important curvature properties is conformal flatness. The conformal (Weyl) curvature tensor is a measure of the curvature of spacetime and differs from the semi-Riemannian curvature tensor. It is the traceless component of the Riemannian tensor which has the same symmetries as the Riemannian tensor. The most important of its special property that it is invariant under conformal changes to the metric. Namely, if  $g^* = kg$  for some positive scalar functions  $k$ , then the Weyl tensor satisfies the equation  $W^* = W$ . In other words, it is called conformal tensor. Weyl constructed a generalized curvature tensor of type  $(1, 3)$  on a semi-Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric; for this reason he called it the conformal curvature tensor of the metric. The Weyl conformal curvature tensor is defined by

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y], \quad (1.3)$$

for all  $X, Y, Z \in T(M)$ , where  $R$  is the curvature tensor,  $S$  is the Ricci tensor and  $r = \text{tr}(S)$  is scalar curvature [15].

A necessary condition for a semi-Riemannian manifold to be conformally flat is that the Weyl curvature tensor vanish. The Weyl tensor vanishes identically for 2 dimensional case. In dimensions  $\geq 4$ , it is generally nonzero. If the Weyl tensor vanishes in dimensions  $\geq 4$ , then the metric is locally conformally flat. So there exists a local coordinate system in which the metric is proportional to a constant tensor. For the dimensions greater than 3, this condition is sufficient as well. But in dimension 3 the vanishing of the equation  $c = 0$ , that is,

$$c(X, Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{2(2n-1)}[(\nabla_X r)Y - (\nabla_Y r)X],$$

is a necessary and sufficient condition for the semi-Riemannian manifold being conformally flat, where  $c$  is the divergence operator of  $C$ , for all vector fields  $X$  and  $Y$  on  $M$ . It should be noted that if the manifold is conformally flat and of dimension greater than 3, then  $C = 0$  implies  $c = 0$  [15].

The concircular curvature tensor  $\bar{C}$  of a  $(2n+1)$ -dimensional manifold is defined by

$$\bar{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y] \quad (1.4)$$

for all  $X, Y, Z \in T(M)$ , where  $R$  is the curvature tensor and  $r = \text{tr}(S)$  is scalar curvature [14], [16]. For  $n \geq 1$ ,  $M$  is concircular flat if and only if the well known concircular curvature tensor  $\bar{C}$  vanishes.

The paper is organized in the following way.

Section 2 is preliminary section, we remember the class of almost paracontact metric manifolds which defined by

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi, \quad (1.5)$$

where  $\alpha$  is a function. These manifolds are called almost  $\alpha$ -paracosymplectic [8]. They contain properly almost paracosymplectic,  $\alpha = 0$ , and almost  $\alpha$ -para-Kenmotsu,  $\alpha = \text{const.} \neq 0$  manifolds. In this section, we remember basic properties of such manifolds.

Section 3 and 4 are devoted to properties of almost  $\alpha$ -para-cosymplectic and almost  $\alpha$ -para-Kenmotsu manifolds. Section 5 devoted to almost  $\alpha$ -para-Kenmotsu manifolds with the  $\eta$ -parallel tensor field  $\phi h$ .

In Section 6, 7 and 8 we study, respectively, projectively flat, conformally flat and concircularly flat almost  $\alpha$ -para-Kenmotsu manifolds (with the  $\eta$ -parallel tensor field  $\phi h$ ). Finally, we present an example to verify our results.

## 2. PRELIMINARIES

Let  $M$  be a  $(2n+1)$ -dimensional differentiable manifold and  $\phi$  is a  $(1, 1)$  tensor field,  $\xi$  is a vector field and  $\eta$  is a one-form on  $M$ . Then  $(\phi, \xi, \eta)$  is called an almost paracontact structure on  $M$  if

- (i)  $\phi^2 = Id - \eta \otimes \xi$ ,  $\eta(\xi) = 1$ ,
- (ii) the tensor field  $\phi$  induces an almost paracomplex structure on the distribution  $D = \ker \eta$ , that is the eigendistributions  $D^\pm$ , corresponding to the eigenvalues  $\pm 1$ , have equal dimensions,  $\dim D^+ = \dim D^- = n$ .

The manifold  $M$  is said to be an almost paracontact manifold if it is endowed with an almost paracontact structure [17].

Let  $M$  be an almost paracontact manifold.  $M$  will be called an almost paracontact metric manifold if it is additionally endowed with a pseudo-Riemannian metric  $g$  of a signature  $(n+1, n)$ , i.e.

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \quad (2.1)$$

For such manifold, we have

$$\eta(X) = g(X, \xi), \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0. \quad (2.2)$$

Moreover, we can define a skew-symmetric tensor field (a 2-form)  $\Phi$  by

$$\Phi(X, Y) = g(\phi X, Y), \quad (2.3)$$

usually called fundamental form. For an almost  $\alpha$ -paracosymplectic manifold, there exists an orthogonal basis  $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$  such that  $g(X_i, X_j) = \delta_{ij}$ ,  $g(Y_i, Y_j) = -\delta_{ij}$  and  $Y_i = \phi X_i$ , for any  $i, j \in \{1, \dots, n\}$ . Such basis is called a  $\phi$ -basis.

An almost paracontact metric manifold is called Einstein if its Ricci tensor  $S$  satisfies the condition

$$S(X, Y) = ag(X, Y).$$

On an almost paracontact manifold, one defines the (1, 2)-tensor field  $N^{(1)}$  by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi,$$

where  $[\phi, \phi]$  is the Nijenhuis torsion of  $\phi$

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

If  $N^{(1)}$  vanishes identically, then the almost paracontact manifold (structure) is said to be normal [17]. The normality condition says that the almost paracomplex structure  $J$  defined on  $M \times \mathbb{R}$

$$J(X, \lambda \frac{d}{dt}) = (\phi X + \lambda \xi, \eta(X) \frac{d}{dt}),$$

is integrable.

### 3. ALMOST $\alpha$ -PARACOSYMPLECTIC MANIFOLDS

An almost paracontact metric manifold  $M^{2n+1}$ , with a structure  $(\phi, \xi, \eta, g)$  is said to be an almost  $\alpha$ -paracosymplectic manifold [8], if

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi, \quad (3.1)$$

where  $\alpha$  may be a constant or function on  $M$ .

For a particular choices of the function  $\alpha$  we have the following subclasses,

- almost  $\alpha$ -para-Kenmotsu manifolds,  $\alpha = \text{const.} \neq 0$ ,
- almost paracosymplectic manifolds,  $\alpha = 0$ .

If additionally normality condition is fulfilled, then manifolds are called  $\alpha$ -para-Kenmotsu or paracosymplectic, resp.

**Definition 1.** [8] For an almost  $\alpha$ -paracosymplectic manifold, define the  $(1, 1)$ -tensor field  $\mathcal{A}$  by

$$\mathcal{A}X = -\nabla_X \xi. \tag{3.2}$$

**Proposition 1.** [8] For an almost  $\alpha$ -paracosymplectic manifold  $M^{2n+1}$ , we have

$$\begin{aligned} i) \mathcal{L}_\xi \eta &= 0, \quad ii) g(\mathcal{A}X, Y) = g(X, \mathcal{A}Y), \quad iii) \mathcal{A}\xi = 0, \\ iv) \mathcal{L}_\xi \Phi &= 2\alpha\Phi, \quad v) (\mathcal{L}_\xi g)(X, Y) = -2g(\mathcal{A}X, Y), \\ vi) \eta(\mathcal{A}X) &= 0, \quad vii) d\alpha = f\eta \text{ if } n \geq 2 \end{aligned} \tag{3.3}$$

where  $\mathcal{L}$  indicates the operator of the Lie differentiation,  $X, Y$  are arbitrary vector fields on  $M^{2n+1}$  and  $f = i_\xi d\alpha$ .

**Proposition 2.** [8] For an almost  $\alpha$ -paracosymplectic manifold, we have

$$\mathcal{A}\phi + \phi\mathcal{A} = -2\alpha\phi, \tag{3.4}$$

$$\nabla_\xi \phi = 0. \tag{3.5}$$

Let define  $h = \frac{1}{2}\mathcal{L}_\xi \phi$ . In the following proposition we establish some properties of the tensor field  $h$ .

**Proposition 3.** [8] For an almost  $\alpha$ -paracosymplectic manifold, we have the following relations

$$g(hX, Y) = g(X, hY), \tag{3.6}$$

$$h \circ \phi + \phi \circ h = 0, \tag{3.7}$$

$$h\xi = 0, \tag{3.8}$$

$$\nabla \xi = \alpha\phi^2 + \phi \circ h = -\mathcal{A}. \tag{3.9}$$

**Corollary 1.** [8] All the above Propositions imply the following formulas for the traces

$$\begin{aligned} tr(\mathcal{A}\phi) &= tr(\phi\mathcal{A}) = 0, \quad tr(h\phi) = tr(\phi h) = 0, \\ tr(\mathcal{A}) &= -2\alpha n, \quad tr(h) = 0. \end{aligned} \tag{3.10}$$

**Proposition 4.** [8] For an almost  $\alpha$ -paracosymplectic manifold, we have

$$\phi(\nabla_{\phi X} \phi)Y + (\nabla_X \phi)Y = -2\alpha\eta(Y)\phi X + g(\alpha\phi X + hX, Y)\xi. \tag{3.11}$$

**Theorem 1.** [8] *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost  $\alpha$ -paracosymplectic manifold. Then, for any  $X, Y \in \chi(M^{2n+1})$ ,*

$$\begin{aligned} R(X, Y)\xi &= d\alpha(X)(Y - \eta(Y)\xi) - d\alpha(Y)(X - \eta(X)\xi) \\ &\quad + \alpha\eta(X)(\alpha Y + \phi hY) - \alpha\eta(Y)(\alpha X + \phi hX) \\ &\quad + (\nabla_X \phi h)Y - (\nabla_Y \phi h)X. \end{aligned} \quad (3.12)$$

#### 4. ALMOST $\alpha$ -PARA-KENMOTSU MANIFOLDS

In this section, we give curvature properties of an almost  $\alpha$ -para-Kenmotsu manifold.

**Theorem 2.** [8] *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost  $\alpha$ -para-Kenmotsu manifold. Then, for any  $X, Y \in \chi(M^{2n+1})$ ,*

$$R(X, Y)\xi = \alpha\eta(X)(\alpha Y + \phi hY) - \alpha\eta(Y)(\alpha X + \phi hX) + (\nabla_X \phi h)Y - (\nabla_Y \phi h)X. \quad (4.1)$$

**Theorem 3.** [8] *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost  $\alpha$ -para-Kenmotsu manifold. Then, for any  $X \in \chi(M^{2n+1})$  we have*

$$R(X, \xi)\xi = lX = -\alpha^2\phi^2X - 2\alpha\phi hX + h^2X - \phi(\nabla_\xi h)X, \quad (4.2)$$

$$(\nabla_\xi h)X = -\alpha^2\phi X - 2\alpha hX + \phi h^2X - \phi R(X, \xi)\xi, \quad (4.3)$$

$$\frac{1}{2}(R(\xi, X)\xi + \phi R(\xi, \phi X)\xi) = \alpha^2\phi^2X - h^2X, \quad (4.4)$$

$$S(X, \xi) = -2n\alpha^2\eta(X) + g(\operatorname{div}(\phi h), X), \quad (4.5)$$

$$S(\xi, \xi) = -2n\alpha^2 + \operatorname{tr}h^2. \quad (4.6)$$

#### 5. ALMOST $\alpha$ -PARA-KENMOTSU MANIFOLDS WITH THE $\eta$ -PARALLEL TENSOR FIELD $\phi h$

For any vector field  $X$  on  $M^{2n+1}$ , we can take  $X = X^T + \eta(X)\xi$ ,  $X^T$  is tangentially part of  $X$  and  $\eta(X)\xi$  the normal part of  $X$ . We say that any symmetric  $(1, 1)$ -type tensor field  $B$  on a semi-Riemannian manifold  $(M, g)$  is said to be a  $\eta$ -parallel tensor if it satisfies the equation

$$g((\nabla_{X^T} B)Y^T, Z^T) = 0,$$

for all tangent vectors  $X^T, Y^T, Z^T$  orthogonal to  $\xi$  [2].

**Proposition 5.** *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost  $\alpha$ -para-Kenmotsu manifold. If the tensor field  $\phi h$  is  $\eta$ -parallel, then we have*

$$\begin{aligned} (\nabla_X \phi h)Y &= \eta(X)[-lY - \alpha^2\phi^2Y - 2\alpha\phi hY + h^2Y] \\ &\quad - \eta(Y)[\alpha\phi hX - h^2X] - g(Y, \alpha\phi hX - h^2X)\xi, \end{aligned} \quad (5.1)$$

for all vector fields  $X, Y$  on  $M$ .

*Proof.* If we suppose that  $\phi h$  is  $\eta$ -parallel, we have

$$\begin{aligned} 0 &= g((\nabla_{X^T} \phi h)Y^T, Z^T) = 0 \\ 0 &= g((\nabla_{X-\eta(X)\xi} \phi h)Y - \eta(Y)\xi, Z - \eta(Z)\xi) \\ 0 &= g((\nabla_X \phi h)Y, Z) - \eta(X)g((\nabla_\xi \phi h)Y, Z) - \eta(Y)g((\nabla_X \phi h)\xi, Z) \\ &\quad - \eta(Z)g((\nabla_X \phi h)Y, \xi) + \eta(X)\eta(Y)g((\nabla_\xi \phi h)\xi, Z) + \eta(Y)\eta(Z)g((\nabla_X \phi h)\xi, \xi) \\ &\quad + \eta(Z)\eta(X)g((\nabla_\xi \phi h)Y, \xi) - \eta(X)\eta(Y)\eta(Z)g((\nabla_\xi \phi h)\xi, \xi), \end{aligned}$$

for all vector fields  $X, Y$  on  $M$ . After some calculations, we get

$$0 = g((\nabla_X \phi h)Y, Z) - \eta(Z)g((\nabla_X \phi h)Y, \xi) - \eta(X)g((\nabla_\xi \phi h)Y, Z) - \eta(Y)g((\nabla_X \phi h)\xi, Z).$$

By (3.9), we obtain

$$(\nabla_X \phi h)Y = \eta(X)(\nabla_\xi \phi h)Y - \eta(Y)(\alpha \phi h X - h^2 X) - g(Y, \alpha \phi h X - h^2 X)\xi.$$

Using the fact that  $(\nabla_\xi \phi h)Y = \phi(\nabla_\xi h)Y$  and (4.3), we have the requested equation.  $\square$

**Proposition 6.** *An almost  $\alpha$ -para-Kenmotsu manifold with the  $\eta$ -parallel tensor field  $\phi h$  satisfies the following relation*

$$R(X, Y)\xi = \eta(Y)lX - \eta(X)lY, \tag{5.2}$$

where  $l = R(\cdot, \xi)\xi$  is the Jacobi operator with respect to the characteristic vector field  $\xi$ .

*Proof.* With the help of the equations (4.1) and (5.1), we get (5.2).  $\square$

**Theorem 4.** *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an almost  $\alpha$ -para-Kenmotsu manifold. If the tensor field  $\phi h$  is  $\eta$ -parallel, then  $\xi$  is the eigenvector of Ricci operator on  $M^{2n+1}$ .*

*Proof.* First, if we take the inner product of (5.2) with  $W$ , we have

$$g(R(X, Y)\xi, W) = \eta(Y)g(lX, W) - \eta(X)g(lY, W),$$

and then replacing  $X, W$  by  $e_i$  in the last equation and taking summation over  $i$  (Let  $\{e_1, e_2, \dots, e_{2n}, \xi\}$  be an  $\phi$ -basis of the tangent space at any point of the manifold), we find

$$\sum_{i=1}^{2n+1} \varepsilon_i g(R(e_i, Y)\xi, e_i) = \sum_{i=1}^{2n+1} \varepsilon_i [\eta(Y)g(le_i, e_i) - \eta(e_i)g(lY, e_i)]$$

for all vector fields  $X, Y$  on  $M$ . From the last equation, one can easily get

$$S(Y, \xi) = \eta(Y)tr(l) \tag{5.3}$$

On the other hand, (5.3) can be written as

$$Q\xi = tr(l)\xi.$$

So, this ends the proof.  $\square$

Using Theorem 9 and Theorem 13 of [8], we can give following corollary.

**Corollary 2.** *Let  $(M^3, \phi, \xi, \eta, g)$  be an almost  $\alpha$ -para-Kenmotsu manifold. If the tensor field  $\phi h$  is  $\eta$ -parallel then an almost  $\alpha$ -paracosymplectic  $(\kappa, \mu, \nu)$ -manifold always exist on every open and dense subset of  $M$ .*

6. PROJECTIVELY FLAT ALMOST  $\alpha$ -PARA-KENMOTSU MANIFOLDS (WITH THE  $\eta$ -PARALLEL TENSOR FIELD  $\phi h$ )

**Theorem 5.** *A projectively flat almost  $\alpha$ -para-Kenmotsu manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$  has a scalar curvature*

$$r = -2ntr(\phi(\nabla_\xi h)) + S(\xi, \xi)(2n + 1). \quad (6.1)$$

*Proof.* Let us suppose that almost  $\alpha$ -para-Kenmotsu manifold is projectively flat. If we take the inner product of (1.2) with  $W$ , we get

$$g(R(X, Y)Z, W) = \frac{1}{2n}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)].$$

Replacing  $W, X$  by  $\xi$  in the last equation and using (4.2), (4.5), we get

$$\begin{aligned} S(Y, Z) = & 2n \left( -\alpha^2 g(Y, Z) + 2\alpha g(\phi Y, hZ) + g(hZ, hY) \right. \\ & \left. + g((\nabla_\xi h)Z, \phi Y) + \frac{1}{2n} \eta(Y)g(\operatorname{div}(\phi h), Z) \right). \end{aligned} \quad (6.2)$$

Considering the  $\phi$ -basis and putting  $Y = Z = e_i$  in (6.2), we obtain

$$\begin{aligned} \sum_{i=1}^{2n+1} \varepsilon_i S(e_i, e_i) &= \sum_{i=1}^{2n+1} \varepsilon_i 2n \left( -\alpha^2 g(e_i, e_i) + 2\alpha g(\phi e_i, h e_i) + g(h e_i, h e_i) \right. \\ & \quad \left. + g((\nabla_\xi h)e_i, \phi e_i) + \frac{1}{2n} \eta(e_i)g(\operatorname{div}(\phi h), e_i) \right) \\ r &= 2n(-\alpha^2(2n + 1) - 2\alpha \operatorname{tr}(\phi h) + \operatorname{tr}(h^2) - \operatorname{tr}(\phi(\nabla_\xi h))) \\ & \quad + \frac{1}{2n} (g(\operatorname{div}(\phi h), \xi)). \end{aligned}$$

Now, using (3.10), (4.5) and (4.6), we get the requested equation.  $\square$

**Theorem 6.** *A projectively flat  $\alpha$ -para-Kenmotsu manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$  is an Einstein manifold.*

*Proof.* If we take  $h = 0$  in the proof of Theorem 5, we obtain  $S(Y, Z) = -2n\alpha^2 g(Y, Z)$ . This means manifold is Einstein.  $\square$

**Theorem 7.** *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an projectively flat almost  $\alpha$ -para-Kenmotsu manifold with the  $\eta$ -parallel tensor field  $\phi h$ . Then  $r = \operatorname{tr}(l)(2n + 1)$ .*

*Proof.* Let us suppose that  $(M^{2n+1}, \phi, \xi, \eta, g)$  is projectively flat almost  $\alpha$ -para-Kenmotsu manifold with the  $\eta$ -parallel tensor field  $\phi h$ . If we take the inner product of (1.2) with  $W$ , we get

$$g(R(X, Y)Z, W) = \frac{1}{2n}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)].$$

By setting  $W = X = \xi$  in the last equation and using Proposition 6 and Theorem 4, we obtain

$$\begin{aligned} g(R(Y, \xi)\xi, Z) &= \frac{1}{2n}[S(Y, Z) - \eta(Y)S(\xi, Z)] \\ g(lY, Z) &= \frac{1}{2n}[S(Y, Z) - \eta(Y)\eta(Z)tr(l)]. \end{aligned}$$

If we set  $Y = Z = e_i$  in the last equation, we end the proof. □

7. CONFORMALLY FLAT ALMOST  $\alpha$ -PARA-KENMOTSU MANIFOLDS

**Theorem 8.** *A conformally flat almost  $\alpha$ -para-Kenmotsu manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$  satisfies the following*

$$0 = tr(\phi(\nabla_\xi h)). \tag{7.1}$$

*Proof.* Let us suppose that almost  $\alpha$ -para-Kenmotsu manifold is conformally flat. If we take the inner product of (1.3) with  $W$ , we get

$$\begin{aligned} g(R(X, Y)Z, W) &= \frac{1}{2n-1}[g(Y, Z)g(QX, W) - g(X, Z)g(QY, W) \\ &\quad + S(Y, Z)g(X, W) - S(X, Z)g(Y, W)] \\ &\quad - \frac{r}{2n(2n-1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned}$$

By setting  $W = X = \xi$  in the last equation and using (4.2), (4.5) and (4.6), we obtain

$$\begin{aligned} S(Y, Z) &= (2n-1)(-\alpha^2g(Y, Z) + \alpha^2\eta(Y)\eta(Z) - 2\alpha g(\phi hY, Z) + g(h^2Y, Z) \\ &\quad - g(\phi(\nabla_\xi h)Y, Z)) - g(Y, Z)(-2n\alpha^2 + tr(h^2) + \eta(Z)(-2n\alpha^2\eta(Y) \\ &\quad + g(\text{div}(\phi h), Y) + \eta(Y)(-2n\alpha^2\eta(Z) + g(\text{div}(\phi h), Z) \\ &\quad + \frac{r}{2n}(g(Y, Z) - \eta(Y)\eta(Z))). \end{aligned} \tag{7.2}$$

Considering the  $\phi$ -basis and putting  $Y = Z = e_i$  in (7.2), we get

$$\begin{aligned} \sum_{i=1}^{2n+1} \varepsilon_i S(e_i, e_i) &= r \\ &= \sum_{i=1}^{2n+1} \varepsilon_i \left\{ (2n-1)(-\alpha^2 g(e_i, e_i) + \alpha^2 \eta(e_i)\eta(e_i) - 2\alpha g(\phi h e_i, e_i)) \right. \\ &\quad + g(h^2 e_i, e_i) - g(\phi(\nabla_\xi h)e_i, e_i) \\ &\quad - g(e_i, e_i)(-2n\alpha^2 + \text{tr}(h^2) + \eta(e_i)(-2n\alpha^2 \eta(e_i) + (\text{div}(\phi h), e_i) \\ &\quad \left. + \eta(e_i)(-2n\alpha^2 \eta(e_i) + (\text{div}(\phi h), e_i) + \frac{r}{2n}(g(e_i, e_i) - \eta(e_i)\eta(e_i))) \right\}. \end{aligned}$$

Then by (3.10), (4.5) and (4.6)), we obtain

$$r = \text{tr}(\phi(\nabla_\xi h)).$$

□

#### 8. CONCIRCULARLY FLAT ALMOST $\alpha$ -PARA-KENMOTSU MANIFOLDS (WITH THE $\eta$ -PARALLEL TENSOR FIELD $\phi h$ )

**Theorem 9.** *A concircularly flat almost  $\alpha$ -para-Kenmotsu manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$  has a scalar curvature*

$$r = (2n+1)(S(\xi, \xi) - \text{tr}(\phi(\nabla_\xi h))). \quad (8.1)$$

*Proof.* Let us suppose that almost  $\alpha$ -para-Kenmotsu manifold is concircularly flat. If we take the inner product of (1.4) with  $W$ , we get

$$g(R(X, Y)Z, W) = \frac{r}{2n(2n+1)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

By setting  $W = X = \xi$  in the last equation and using (4.2), we obtain

$$\begin{aligned} \alpha^2 g(\phi Y, \phi Z) - 2\alpha g(\phi h Y, Z) + g(h^2 Y, Z) - g(\phi(\nabla_\xi h)Y, Z) \\ = \frac{r}{2n(2n+1)} (g(Y, Z) - \eta(Y)\eta(Z)). \end{aligned} \quad (8.2)$$

Considering the  $\phi$ -basis and putting  $Y = Z = e_i$  in (8.2), we get

$$\begin{aligned} \sum_{i=1}^{2n+1} \varepsilon_i \left( -\alpha^2 g(e_i, e_i) + \alpha^2 \eta(e_i)\eta(e_i) - 2\alpha g(\phi h e_i, e_i) + g(h^2 e_i, e_i) \right. \\ \left. - g(\phi(\nabla_\xi h)e_i, e_i) \right) \\ = \sum_{i=1}^{2n+1} \varepsilon_i \left( \frac{r}{2n(2n+1)} (g(e_i, e_i) - \eta(e_i)\eta(e_i)) \right). \end{aligned}$$

Then by (3.10), (4.5) and (4.6), we obtain (8.1). □

**Theorem 10.** *A concircularly flat  $\alpha$ -para-Kenmotsu manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$  has a scalar curvature*

$$r = -2\alpha^2 n(2n + 1). \tag{8.3}$$

*Proof.* If we take  $h = 0$  in the proof of Theorem 9, we obtain the requested equation.  $\square$

**Theorem 11.** *Let  $(M^{2n+1}, \phi, \xi, \eta, g)$  be an concircularly flat almost  $\alpha$ -para-Kenmotsu manifold with the  $\eta$ -parallel tensor field  $\phi h$ . Then  $r = \text{tr}(l)(2n + 1)$ .*

*Proof.* Let us suppose that  $(M^{2n+1}, \phi, \xi, \eta, g)$  is concircularly flat almost  $\alpha$ -para-Kenmotsu manifold with the  $\eta$ -parallel tensor field  $\phi h$ . If we take the inner product of (1.4) with  $W$ , we get

$$g(R(X, Y)Z, W) = \frac{r}{2n(2n + 1)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].$$

By setting  $W = X = \xi$  in the last equation and using Proposition 6 and Theorem 4, we obtain

$$g(R(Y, \xi)\xi, Z) = g(lY, Z) = \frac{r}{2n(2n + 1)} [g(Y, Z) - \eta(Y)\eta(Z)].$$

If we set  $Y = Z = e_i$  in the last equation, we end the proof.  $\square$

9. EXAMPLE

Now, we will give an example of a 3-dimensional para-Kenmotsu manifold ( $\alpha = 1$ ).

**Example 1.** *We consider the 3-dimensional manifold*

$$M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$$

*and the vector fields*

$$X = \frac{\partial}{\partial x}, \quad \phi X = \frac{\partial}{\partial y}, \quad \xi = (x + 2y) \frac{\partial}{\partial x} + (2x + y) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

*The 1-form  $\eta = dz$  defines an almost paracontact structure on  $M$  with characteristic vector field  $\xi = (x + 2y) \frac{\partial}{\partial x} + (2x + y) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ . Let  $g, \phi$  be the pseudo-Riemannian metric and the  $(1, 1)$ -tensor field given by*

$$g = \begin{pmatrix} 1 & 0 & -\frac{1}{2}(x + 2y) \\ 0 & -1 & \frac{1}{2}(2x + y) \\ -\frac{1}{2}(x + 2y) & \frac{1}{2}(2x + y) & 1 - (2x + y)^2 + (x + 2y)^2 \end{pmatrix},$$

$$\phi = \begin{pmatrix} 0 & 1 & -(2x + y) \\ 1 & 0 & -(x + 2y) \\ 0 & 0 & 0 \end{pmatrix},$$

*with respect to the basis  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ .*

Using (3.9) we have

$$\begin{aligned}\nabla_X X &= -\xi, & \nabla_{\phi X} X &= 0, & \nabla_\xi X &= -2\phi X, \\ \nabla_X \phi X &= 0, & \nabla_{\phi X} \phi X &= \xi, & \nabla_\xi \phi X &= -2X, \\ \nabla_X \xi &= X, & \nabla_{\phi X} \xi &= \phi X, & \nabla_\xi \xi &= 0.\end{aligned}$$

for  $\alpha = 1$ . Hence the manifold is a para-Kenmotsu manifold. One can easily compute,

$$\begin{aligned}R(X, \phi X)\xi &= 0, & R(\phi X, \xi)\xi &= -\phi X, & R(X, \xi)\xi &= -X, \\ R(X, \phi X)\phi X &= X, & R(\phi X, \xi)\phi X &= -\xi, & R(X, \xi)\phi X &= 0, \\ R(X, \phi X)X &= \phi X, & R(\phi X, \xi)X &= 0, & R(X, \xi)X &= \xi.\end{aligned}\tag{9.1}$$

We have constant scalar curvature as follows,

$$r = S(X, X) - S(\phi X, \phi X) + S(\xi, \xi) = -6.$$

So, we conclude that  $M$  is a three dimensional projectively flat and concircularly flat para-Kenmotsu manifold for  $\alpha = 1$ .

#### REFERENCES

- [1] Bejan, C.L., Almost parahermitian structures on the tangent bundle of an almost para-coHermitian manifold, In: The Proceedings of the Fifth National Seminar of Finsler and Lagrange Spaces (Braşov, 1988), 105–109, Soc. Stiinte Mat. R. S. Romania, Bucharest, 1989.
- [2] Boeckx, E. and Cho, J. T.,  $\eta$ -parallel contact metric spaces, *Differ. Geom. Appl.* 22, (2005), 275-285.
- [3] Buchner, K. and Rosca, R., Variétés para-coKählerian á champ concirculaire horizontale, *C. R. Acad. Sci. Paris* 285 (1977), Ser. A, 723-726.
- [4] Buchner, K. and Rosca, R., Co-isotropic submanifolds of a para-coKählerian manifold with concircular vector field, *J. Geometry*, 25 (1985), 164-177.
- [5] Cappelletti-Montano, B., Küpeli Erken, I. and Murathan, C., Nullity conditions in paracontact geometry, *Diff. Geom. Appl.* 30 (2012), 665-693.
- [6] Dacko, P., On almost para-cosymplectic manifolds, *Tsukuba J. Math.* 28 (2004), 193-213.
- [7] Kaneyuki, S. and Williams, F. L., Almost paracontact and parahodge structures on manifolds, *Nagoya Math. J.* 1985; 99: 173-187.
- [8] Küpeli Erken, I., Dacko, P. and Murathan, C., Almost  $\alpha$ -paracosymplectic manifolds, *J. Geom. Phys.*, 88 (2015) 30-51.
- [9] Öztürk, H., Aktan, N. and Murathan, C., Almost  $\alpha$ -cosymplectic  $(\kappa, \mu, \nu)$ -spaces, Submitted. Available in Arxiv:1007.0527 [math. DG].
- [10] Öztürk, H., Murathan, C., Aktan, N. and Turgut Vanlı, A., Almost  $\alpha$ -cosymplectic f-manifolds, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* 60 no. 1, (2014), 211-226.
- [11] Rosca, R. and Vanhecke, L., Sur une variété presque paracokählerienne munie d'une connexion self-orthogonale involutive, *Ann. Sti. Univ. "Al. I. Cuza" Iaşi* 22 (1976), 49-58.
- [12] Soos, G., Über die geodätischen Abbildungen von Riemannschen Räumen auf projektiv symmetrische Riemannsche Räume, *Acta. Math. Acad. Sci. Hungar.* Tom 9 (1958), 359-361.
- [13] Węlyczko, J., On basic curvature identities for almost (para)contact metric manifolds. Available in Arxiv: 1209.4731 [math. DG].
- [14] Yano, K., Concircular geometry I, Concircular transformations, *Proc. Imp. Acad. Tokyo*, 16 (1940), 195-200.
- [15] Yano, K. and Kon, M., Structures on manifolds, Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, (1984).

- [16] Yano, K. and Bochner, S., Curvature and Betti numbers, Annals of mathematics studies, 32, Princeton University Press, 1953.
- [17] Zamkovoy, S., Canonical connections on paracontact manifolds, *Ann. Glob. Anal. Geom.* 36 (2009), 37-60.

*Current address:* Faculty of Engineering and Natural Sciences, Department of Mathematics, Bursa Technical University, Bursa, TURKEY

*E-mail address:* [irem.erken@btu.edu.tr](mailto:irem.erken@btu.edu.tr)

ORCID Address: <http://orcid.org/0000-0003-4471-3291>