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BINOMIAL-DISCRETE LINDLEY DISTRIBUTION

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Abstract. In this paper, a new discrete distribution called Binomial-Discrete Lindley (BDL) distribution is proposed by compounding the binomial and discrete Lindley distributions. Some properties of the distribution are discussed including the moment generating function, moments and hazard rate function. The estimation of distribution parameter is studied by methods of moments, proportions and maximum likelihood. A simulation study is performed to compare the performance of the different estimates in terms of bias and mean square errors. Automobile claim data applications are also presented to see that the new distribution is useful in modelling data.

1. Introduction

Sometimes in life testing experiments, the life length of a device can not be measured on a continuous scale and the reliability (survival) function is assumed to be a function of a count (discrete) random variable instead of being a function of continuous time random variable. For example, the reliability of a computer is a function of the number of break down of the computer or the reliability of a switching device is a function of the number of times the switch is operated. On the other hand in some cases, if the life length can be measured on a continuous scale, the measurements cannot be recorded with desired sensitivity. In such situations, it is reasonable to consider the observations as coming from a discretized distribution generated from the underlying continuous model. Therefore, discrete distributions are quite meaningful to model life time data in such situations[1], [2].

Recently, many discrete lifetime distributions have been proposed in the statistical literature by discretizing the continuous lifetime distributions. See, for example, [3], [5], [10], [13], [14], [15], [16] and [17].

In this paper, by using methodology of [4] and [8], a new discrete distribution is proposed apart from discretizing continuous distributions. If N and X are two discrete random variables denoting the number of particles entering and leaving an

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attenuator, then [8], showed that the probability mass functions of $p(n)$ and $f(x)$ of these two random variables are connected by the binomial decay transformation

$$P(X=x)=\sum_{n=x}^{\infty}\binom{x}{n}p^{x}(1-p)^{n-x}p(n),\ x=0,1,... \tag{1}$$

which $0\leq p\leq 1$ is the attenuating coefficient which is discussed in [8]. They considered $p(n)$ as a Poisson distribution with parameter $\lambda>0$ and then showed that $P(X=x)$ is Poisson distribution with parameter λp. Recently, some new discrete distributions are proposed in the literature using methodology of [8]. See, for example, [1] and [4].

The rest of the paper is structured as follows. In Section 2, the Binomial-Discrete Lindley distribution is introduced and some properties of the distribution such as moments, moment generating function and hazard rate function are obtained. In Section 3, some estimates of the distribution parameter are discussed by maximum likelihood, moments and proportions methodology. Simulation study is conducted to compare the performance of the different estimates in Section 4. Finally, Section 5 illustrates the application of the proposed distribution in modelling automobile claim frequency data.

2. The distribution and some properties

The probability mass function (pmf) given in (1) can be expressed as

$$P\left(X=x\right)=\sum_{n=x}^{\infty}P\left(X=x\mid N=n\right)P\left(N=n\right),$$

where $X|N=n$ has Binomial $b(n,p)$ distribution. Now, let the distribution of $X|N=n$ is Binomial $b\left(n,p\right)$ distribution and N follows the discrete Lindley distribution with pmf (see [5])

$$p(n)=P(N=n)=\frac{p^{n}}{1-\log p}\left[p\log p+(1-p)\left(1-\log p^{n+1}\right)\right]$$

for $n=0,1,...$ and $0<p<1$. Then the marginal pmf of X is obtained as

$$\begin{aligned}
f\left(x\right) &= \sum_{n=x}^{\infty}P\left(X=x\mid N=n\right)\ P\left(N=n\right),\\
&= \sum_{n=x}^{\infty}\binom{n}{x}p^{x}\left(1-p\right)^{n-x}\frac{p^{n}}{1-\log p}\left[p\log p+(1-p)\left(1-\log p^{n+1}\right)\right]\\
&= \sum_{j=0}^{\infty}\binom{x+j}{x}p^{x}\left(1-p\right)^{j}\frac{p^{x+j}}{1-\log p}\left[p\log p+(1-p)\left(1-\log\left(p^{x+j+1}\right)\right)\right]\\
&= \frac{p^{2x}\left[\left(p^{3}-(1-p)\left(1-p+x\right)\right)\log p+\left(1-p\right)\left(1-p\left(1-p\right)\right)\right]}{\left(1-\log p\right)\left(1-p\left(1-p\right)\right)^{x+2}}, \qquad (2)
\end{aligned}$$

for $x = 0, 1, \ldots$. If X has the pmf (2), then it is called a Binom Discrete Lindley (BDL) random variable and it is denoted by $X \sim BDL(p)$.

The cumulative distribution function(cdf) of X can be obtained as

$$\begin{aligned} F(x) &= \sum_{n=0}^{x} \frac{p^{2n}\left[\left(p^3-(1-p)(1-p+n)\right)\log p+(1-p)(1-p(1-p))\right]}{(1-\log p)(1-p(1-p))^{n+2}} \\ &= 1-\frac{\left\{[1+(-2-x)\log p]\,p^{(2x+2)}+\left(p^{(2x+3)}(1-p)\right)(\log p-1)\right\}\left(p^2-p+1\right)^{-2-x}}{1-\log p} \end{aligned}$$

for $x = 0, 1, \ldots$

Theorem 1. *The pmf in (2) is log-concave.*

Proof. From ([12], [18], [6], [11]), a distribution with pmf $f_X(x)$ is log-concave if

$$f_X(x+1)^2 > f_X(x)\, f_X(x+2) \tag{3}$$

for all $x \geq 0$. Under $p \in (0, 1)$

$$f_X(x+1)^2 - f_X(x)\, f_X(x+2) = \frac{p^{(4x+4)}\left(p^2-p+1\right)^{(-2x-6)}(\log p)^2(p-1)^2}{(\log p-1)^2} > 0$$

for all $x \geq 0$. So (3) is satisfied for pmf (2). $\square$

Figure 1 presents the plots of the $DBL(p)$ mass function for some choices of p. From the log-concavity $f(x)$ in (2), $BDL(p)$ is unimodal.If X has the $BDL(p)$ distribution, then the moment generating function of X is obtained as

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) \\ &= \sum_{x=0}^{\infty} e^{tX}\left\{\frac{p^{2x}\left[\left(p^3-(1-p)(1-p+x)\right)\log p+(1-p)(1-p(1-p))\right]}{(1-\log p)(1-p(1-p))^{x+2}}\right\} \\ &= \frac{\left(-p^3+p^3e^t+p^2-2p+1\right)\log p-(p-1)\left(p^2e^t-p^2+p+1\right)}{(\log p-1)\left(p^2e^t-p^2+p+1\right)^2}. \end{aligned}$$

Using the moment generating function $M_X(t)$, we can obtain the probability generating function of $BDL(p)$ distribution as

$$\begin{aligned} \psi_X(t) &= E(t^X) \\ &= M_X(\log(t)) \\ &= \frac{\left(1+(t-1)p^3+p^2-2p\right)\log p-(p-1)\left((t-1)p^2+p-1\right)}{(\log p-1)\left((t-1)p^2+p-1\right)^2}. \end{aligned}$$

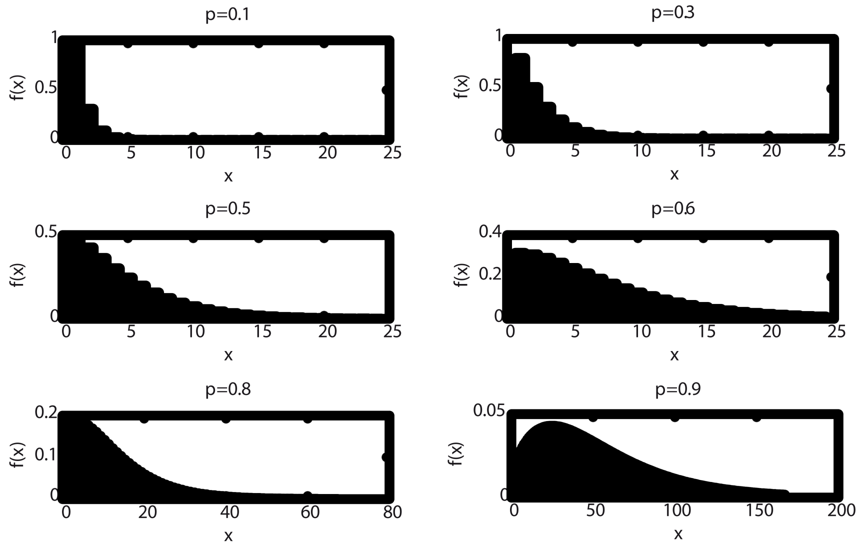

FIGURE 1. Pmf of BDL(p) distribution for some choices of p

The expected value of the $BDL\,(p)$ distribution is obtained as

$$\begin{aligned} E\,(X) &= \sum_{x=0}^{\infty} x \left(\frac{p^{2x}\left[\left(p^3-(1-p)\,(1-p+x)\right)\log p+(1-p)\,(1-p\,(1-p))\right]}{(1-\log p)\,(1-p\,(1-p))^{x+2}} \right) \\ &= \frac{(p\log p-p-2\log p+1)\,p^2}{(1-\log p)\,(p-1)^2}. \end{aligned} \tag{4}$$

Note that the expressions for higher moments, skewness and kurtosis of the BDL(p) distribution are too cumbersome and they are not reported here. Table 1 presents the skewness and kurtosis of the BDL(p) distribution for different values of the parameter p. From Table 1 and Figure 1, it can be easily seen that skewness and kurtosis are inversely proportional to p.

Table 1. Skewness and kurtosis of the BDL distribution for different values of the parameter p.

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Skewness	7.2509	3.6302	2.5036	1.9996	1.7384	1.5914	1.5044	1.4537	1.4237
Kurtosis	56.8091	17.4025	10.4588	8.1451	7.1219	6.5914	6.2913	6.1186	6.0281

Theorem 2. *If $X \sim BDL\,(p)$, then $Var\,(X) \geq E\,(X)$ for $p \in (0,1)$.*

Proof. Let us consider following identity:

$$\begin{aligned} Var\,(X) - E\,(X) &= E\left(X^2\right) - E\,(X)\,(1 + E\,(X)) \\ &= \frac{\left[-2p^3 + 2p^3 \log p - 7p^2 \log p + 3p^2 - 2p + 3p \log p - 2 \log p + 1\right] p^2}{(p-1)^3 (\log p - 1)} \\ &\quad - \frac{(p \log p - p - 2 \log p + 1)\, p^2}{(1 - \log p)\,(p-1)^2} \left\{ 1 + \frac{(p \log p - p - 2 \log p + 1)\, p^2}{(1 - \log p)\,(p-1)^2} \right\}. \end{aligned}$$

After some simple algebras, we have

$$Var\,(X) - E\,(X) \geq 0 \Longleftrightarrow \underbrace{\left(p^2 - 4p + 2\right)(\log p)^2 + (p-1)^2}_{f_1(p)} - \underbrace{\left(2p^2 - 6p + 4\right) \log p}_{f_2(p)} \geq 0.$$

It is clear that $f_2\,(p) < 0$ for $p \in (0,1)$. In order to show $f_1\,(p) - f_2\,(p) \geq 0$ it is enough to show that $f_1\,(p) \geq 0$ for $p \in (0,1)$. For $p \in (0,1)$, we can write

$$\left(2p^2 - 4p + 2\right) = \left(p^2 - 4p + 3\right) - \left(1 - p^2\right) \geq 0. \tag{5}$$

On the other hand, using the well-known relation

$$p < \left(\frac{p-1}{\log p}\right)$$

in (5), we can obtain the following inequalities

$$\left(p^2 - 4p + 3\right) - \left\{1 - \left(\frac{p-1}{\log p}\right)^2\right\} = \frac{\left(p^2 - 4p + 3\right) \log^2 p - \log^2 p + (p-1)^2}{\log^2 p} \geq 0 \tag{6}$$

From (6), we conclude that

$$f_1\,(p) = \left(p^2 - 4p + 2\right) \log^2 p + (p-1)^2 \geq 0.$$

Consequently, $f_1(p) \geq 0$ for $p \in (0,1)$ and the proof is completed. As a consequence of Theorem 2, BDL is overdispersed. □

The hazard (failure) rate function of the discrete random variable $X \sim BDL\,(p)$ is defined by

$$h\,(x) = P\,(X = x \mid X \geq x) = \frac{P\,(X = x)}{P\,(X \geq x)}, \tag{7}$$

provided $P\,(X \geq x) > 0$. Eq (7) may be considered as the classical discrete hazard rate function. From Eq. (2) the hazard rate function of $BDL\,(p)$ distribution is

$$h(x) = \frac{\log p \left(p^3 - p^2 + px + 2p - 1 - x\right) - p^3 + 2p^2 - 2p + 1}{\left(p^2 - p + 1\right)\left(-p^2 + p^2 \log p + p - p \log p - 1 + \log p + x \log p\right)}$$

From Theorem 1, the pmf is log-concave and hence the BDL distribution has in-

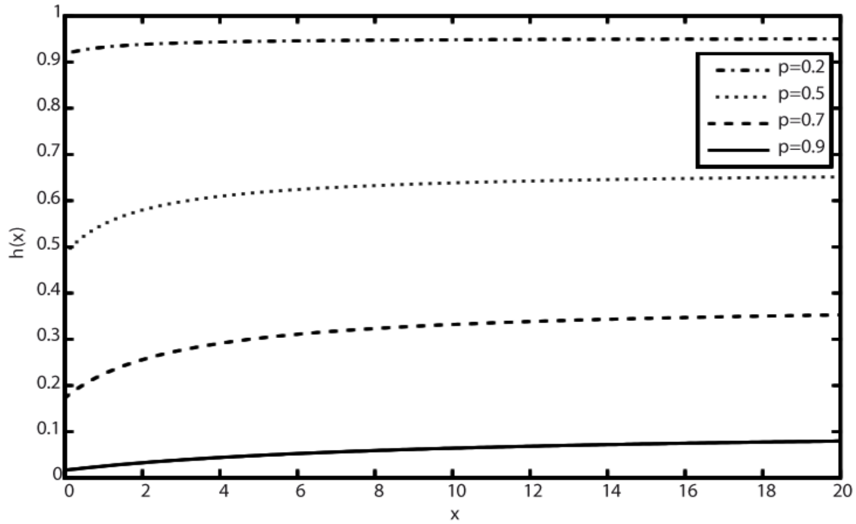

FIGURE 2. Hazard rate function for different values of p

creasing failure rate. Figure 2 provides the hazard rate function of BDL(p) distribution for selected values of p.

3. POINT ESTIMATIONS

In this section, we discuss the estimation of unknown parameter p of the BDL distribution by maximum likelihood, proportion and moment methodology.

3.1. **Maximum likelihood estimation.** Let $X_1, X_2, \ldots, X_n$ be a random sample, with observed values $x_1, x_2, \ldots, x_n$ from $BDL\,(p)$. The likelihood and log-likelihood functions are given respectively by

$$L(p) = \prod_{i=1}^{n} \frac{p^{2x_i}\left[\left(p^3 - (1-p)\,(1-p+x_i)\right)\log p + (1-p)\,(1-p\,(1-p))\right]}{(1-\log p)\,(1-p\,(1-p))^{x_i+2}}$$

and

$$\begin{aligned}\ell\,(p) \quad &= \quad \sum_{i=1}^{n} \log\left(\left(p^2 - p^3 + p\,(-x_i-2) + x_i + 1\right)\log p + p^3 - 2p^2 + 2p - 1\right) \\ &\quad + (-2-x_i)\log\left(p^2 - p + 1\right) + 2x_i \log p - \log\left(\log p - 1\right),\end{aligned}$$

Thus, the likelihood equation is obtained as

$$\begin{aligned}
&\frac{\partial \log \ell(p)}{\partial p} \\
= \quad & \sum_{i=1}^{n} \frac{\left(-3p^2+2p-2-x_i\right) p \log p + 3p^3 - 4p^2 + 2p}{\left(p^2-p^3+p\left(-x_i-2\right)+x_i+1\right) p \log p + p^4 - 2p^3 + 2p^2 - p} \\
& + \frac{\left(p^2-p^3+p\left(-x_i-2\right)+x_i+1\right)}{\left(p^2-p^3+p\left(-x_i-2\right)+x_i+1\right) p \log p + p^4 - 2p^3 + 2p^2 - p} \\
& + \frac{\left(-x_i-2\right)\left(2p-1\right)}{p^2-p+1} + \frac{2x_i}{p} - \frac{1}{p\left(-1+\log p\right)} = 0. \qquad (8)
\end{aligned}$$

The maximum likelihood estimate (MLE) of parameter, i.e., $\hat{p}$, can be achieved by solving Eq (8) using some numerical procedures such as the Newton-Raphson procedure.

3.2. **Method of moments.** Let us $X_1, X_2, \ldots, X_n$ be a random sample from the $BDL(p)$ distribution. To estimate the parameter p by the method of moments (MM), we need to solve the moment equation $E(X) = \overline{X}$, i.e.,

$$\frac{\left(p\log p - p - 2\log p + 1\right) p^2}{\left(1-\log p\right)\left(p-1\right)^2} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad (9)$$

Eq (9) can be solved numerically via Newton-Raphson.

3.3. **Method of proportions.** In this section, method of proportions is adopted to estimate the parameter p from [9]. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $BDL(p)$ distribution. For $i = 0, 1, \ldots, n$, define the indicator function $\nu(.)$ as

$$\nu\left(X_i\right) = \begin{cases} 1, & X_i = 0 \\ 0, & X_i > 0 \end{cases}$$

It is easily seen that $Y = \frac{1}{n}\sum_{i=1}^{n} \nu\left(X_i\right)$ denotes the proportion of $0'$s in the sample. Also, the proportion Y is an unbiased and consistent estimate of the probability

$$f(0) = \frac{\left(p^2-p^3-2p+1\right)\log p + p^3 - 2p^2 + 2p - 1}{\left(-1+\log p\right)\left(p^2-p+1\right)}.$$

Therefore, the method of proportion (MP) estimate of p can be obtained by solving the equation $Y = P(Y = 0)$, i.e.,

$$Y = \frac{\left(p^2-p^3-2p+1\right)\log p + p^3 - 2p^2 + 2p - 1}{\left(-1+\log p\right)\left(p^2-p+1\right)} = h(p) \qquad (10)$$

with respect to p. This equation must be solved by some numerical methods.

4. Simulation Study

In this section, simulation study is performed to compare the performance of estimates given in Section 3. In this simulation, we have generated 10000 random samples with sizes 10, 30, 50 and 100 from the $BDL\left(p\right)$ distribution and then computed the MLE, MM and MP of p. We compared then the performance of these estimators in terms of their biases and mean square errors (MSEs) as follows:

$$Bias_p\left(n\right)=\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{p}_i-p\right)$$

$$MSE_p\left(n\right)=\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{p}_i-p\right)^2$$

In the following, an algorithm is suggested to generate the random sample from $BDL\left(p\right)$ distribution. A random sample from $BDL\left(p\right)$ can be generated by using the following algorithm.

Algorithm 3. *S1. Generate* $U_i\sim$ *Uniform* $(0,1)$*,* $i=1,2,...,n$*,*

S2. Using the probability integral transformation rule $\left(F^{-1}\left(Y_i\right)=U_i\right)$*, generate* Y_i *from the discrete Lindley* $DL(p)$ *distribution. Not that* Y_i *is the root of the equation*

$$1-\tfrac{(\theta(1+Y_i)+1)\exp(-\theta Y_i)}{\theta+1}=U_i \quad , \quad i=1,2,...,n \ ,$$

where Y_i *can be solved by some numerical methods such as the Newton-Raphson method.*

S3. Set $N_i=\lfloor Y_i\rfloor$*,* $i=1,2,...,n$*,where* $\lfloor x\rfloor$ *is the greatest integer less than or equal to* x*.*

S4. For $i=1,2,\ldots,n$ *generate* $X_i\sim$ *Binomial* $\left(N_i,p\right)$*. Then* $X_1,X_2,\ldots,X_n$ *is the required sample from the* $BDL\left(p\right)$ *distribution.*

In Table 2, the biases and MSEs of these estimators are reported. From Table 2, the maximum likelihood and moment estimates have almost identical performance and their MSEs are better than MSE of proportion estimate for all selected parameters setting. Bias of proportion is better the others for small values of p (say $p<0.5$) and worse the the others for large values of p (say $p>0.5$). It should be pointed out that performance of all estimates are the same for large values of sample size of n.

Table 2 . Biases and MSEs of MLE, MM, and LSE estimators for some different values of p.

		MLE		MM		MP	
(p)	n	Bias($\hat{p}$)	Mse($\hat{p}$)	Bias($\hat{p}$)	Mse($\hat{p}$)	Bias($\hat{p}$)	Mse($\hat{p}$)
(0.20)	10	-0.0572	0.0128	-0.0556	0.0125	-0.0510	0.0144
	30	-0.0158	0.0047	-0.0155	0.0046	-0.0136	0.0049
	50	-0.0081	0.0022	-0.0080	0.0022	-0.0069	0.0023
	100	-0.0043	0.0011	-0.0042	0.0010	-0.0035	0.0011
(0.50)	10	-0.0142	0.0051	-0.0142	0.0051	-0.0012	0.0093
	30	-0.0042	0.0015	-0.0042	0.0015	0.0006	0.0028
	50	-0.0027	0.0009	-0.0027	0.0009	0.0008	0.0017
	100	-0.0014	0.0005	-0.0014	0.0005	-0.0002	0.0008
(0.75)	10	-0.0071	0.0018	-0.0071	0.0018	-0.0247	0.0074
	30	-0.0026	0.0005	-0.0026	0.0005	0.0038	0.0033
	50	-0.0021	0.0003	-0.0021	0.0003	0.0035	0.0021
	100	-0.0008	0.0002	-0.0008	0.0002	0.0028	0.0010
(0.95)	10	-0.0020	0.0001	-0.0020	0.0001	0.0225	0.0045
	30	-0.0008	0.0000	-0.0008	0.0000	0.0094	0.0038
	50	-0.0006	0.0000	-0.0006	0.0000	0.0027	0.0030
	100	-0.0004	0.0000	-0.0004	0.0000	-0.0062	0.0017

5. Application

In this section, the number of claims are considered in automobile insurance from five different countries. Six different data sets are given in [7]. The data sets are also used in [19]. All these data which are given in Table 3 present phenomena of over-dispersion, that is, the variance is greater than the mean and, therefore, the Binomial-Discrete Lindley distribution seems to be suitable for fitting them. The BDL, Poisson, Discrete Pareto [10] and Discrete Lindley (DL) [5] models are used to fit the automobile claim frequency data sets.

Table 3. Automobile claim data Willmot (1987).

Number of claims	Country	0	1	2	3	4	5	6	7
Data Set I	Switzerland 1961	103704	14075	1766	255	45	6	2	-
Data Set II	Great-Britain 1968	370412	46545	3935	317	28	3	-	-
Data Set III	Belgium 1958	7840	1317	239	42	14	4	4	1
Data Set IV	Zaire 1974	3719	232	38	7	3	1	-	-
Data Set V	Belgium 1975-76	96978	9240	704	43	9	-	-	-
Data Set VI	Germany 1960	20592	2651	297	41	7	0	1	-

In order to compare the models, we used following criteria: Akaike Information Criterion(AIC), Bayesian Information Criterion (BIC), log-likelihood values which are given in Table 4. From Table 4, BDL distribution gives better fit than the others for the first and last data sets.

Table 4. Results of AIC, BIC and log-likelihood for BDLD and other distributions for automobile claim data sets.

		BDLD	DLD	Poisson	Disc. Pareto
Data Set I	ℓ	-54659.100	-54659.614	-55108.455	-56351.011
	AIC	109320.201	109321.227	110218.910	112704.021
	BIC	109329.895	109330.921	110228.604	112713.715
Data Set II	ℓ	-171198.407	-171196.166	-171373.176	-178321.718
	AIC	342398.813	342394.333	342748.352	356645.437
	BIC	342409.764	342405.283	342759.303	356656.388
Data Set III	ℓ	-5377.784	-5377.510	-5490.780	-5486.714
	AIC	1075.757	1075.702	1098.356	1097.543
	BIC	1076.472	1076.418	1099.072	1098.258
Data Set IV	ℓ	-1217.358	-1217.698	-1246.077	-1186.498
	AIC	2436.717	2437.397	2494.154	2374.997
	BIC	2443.011	2443.691	2500.448	2381.291
Data Set V	ℓ	-36104.236	-36104.217	-36188.254	-37238.158
	AIC	72210.472	72210.435	72378.508	74478.317
	BIC	72220.052	72220.015	72388.088	74487.897
Data Set VI	ℓ	-10228.342	-10228.453	-10297.843	-10551.846
	AIC	20458.684	20458.906	20597.686	21105.693
	BIC	20466.752	20466.975	20605.755	21113.761

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