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COMPLETE SYSTEM OF INVARIANTS OF VECTORS FOR ISOMETRY GROUP IN $n-$DIMENSIONAL UNITARY SPACE

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ABSTRACT. In this study, invariants of vector systems for isometry group are investigated. The complete system of invariants of vectors for isometry group in n-dimensional unitary space is obtained and it is shown that this complete system is a minimal complete system.

1. INTRODUCTION

Geometric invariant theory, as developed by D. Mumford in 1960s (using ideas in classical invariant theory), studies linearizes actions of linear algebraic groups on algebraic varieties and it provides techniques for constructing a categorical quotient within the category of algebraic varieties.

In 1872, F. Klein stated his famous Erlangen Programme: Geometry is the study of invariants with respect to a given transformation group. Klein originally stated his programme for the elementary geometries, but at the beginning of the century the programme was also applied to differential geometry.

One of the important problems in the theory of invariants is finding necessary and sufficient conditions for equivalence of systems of vectors $\{x_1, ..., x_m\}$ and $\{y_1, ..., y_m\}$ under the action of the group suggested. This study presents the conditions for the equivalence of two vector systems for isometry group in the $n-$dimensional unitary space. So, the complete system of invariants of vectors for isometry group in $n-$dimensional unitary space is obtained. In addition, it's also obtained that this system is a minimal complete system.

The invariants of vectors and curves relative to the Euclidean group, affine group and Lorentz group are investigated in [1, 2, 5, 6, 7, 8]. Some applications of the invariant theory are given in [3, 4].

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2. Preliminaries

Definition 1. *Let V be a vector space over the field F. An inner product on V is a function $<,>: V \times V \to F$ with the following properties:*

i. *For all $v \in V$, $< v, v > \geq 0$ and $< v, v >= 0 \iff v = 0$.*
ii. *If $F = \mathbb{C}$, $< u, v >= \overline{< v, u >}$. If $F = \mathbb{R}$, $< u, v >=< v, u >$.*
iii. *For all $u, v, w \in V$ and $r, s \in F$*
$< ru + sv, w >= r < u, v > +s < v, w >$.

A real (or complex) vector space V, together with an inner product, is called a real (or complex) inner product space.

Example 1.

1. *The vector space $\mathbb{R}^n$ is an inner product space under the standard inner product, or dot product, defined by*

$$< (r_1, ..., r_n), (s_1, ..., s_n) >= r_1 s_1 + ... + r_n s_n.$$

The inner product space $\mathbb{R}^n$ is often called $n-$dimensional Euclidean space.

2. *The vector space $\mathbb{C}^n$ is an inner product space under the standard inner product defined by*

$$< (r_1, ..., r_n), (s_1, ..., s_n) >= r_1 \overline{s_1} + ... + r_n \overline{s_n}.$$

This inner product space is often called $n-$dimensional unitary space.

Definition 2. *A transformation $F : \mathbb{C}^n \to \mathbb{C}^n$ is called $\mathbb{C}-$linear ($\mathbb{R}-$linear) operator provided following conditions:*

i. *$F(z_1 + z_2) = F(z_1) + F(z_2)$, for all $z_1, z_2 \in \mathbb{C}^n$.*
ii. *$F(\lambda z) = \lambda F(z)$, for all $z \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ (for all $\lambda \in \mathbb{R}$).*

Proposition 1. *A transformation $H : \mathbb{C}^n \to \mathbb{R}^{2n}$, $H(z) = (Re(z), Im(z))$, where $z = (a_1, ..., a_n) + i(b_1, ..., b_n)$, $i = \sqrt{-1}$, is an $\mathbb{R}-$isomorphism (an invertible $\mathbb{R}-$linear transformation).*
Moreover, the inverse of H is also an $\mathbb{R}-$isomorphism.

Proof. The proof is straightforward. □

Remark 1. *Let $M(\mathbb{R}^{2n}, \mathbb{R})$ be the set of all linear transformations in the vector space $\mathbb{R}^{2n}$. The set $M(\mathbb{R}^{2n}, \mathbb{R})$ is a real vector space with respect to the operations addition of the real linear transformations and multiplying with a real number.*

Let $M(\mathbb{C}^n, \mathbb{R})$ be the set of all linear transformations in the vector space $\mathbb{C}^n$. The set $M(\mathbb{C}^n, \mathbb{R})$ is a real linear complex vector space with respect to the operations addition of the real linear transformations and multiplying with a complex number.

Theorem 1. *Transformation $W : M(\mathbb{C}^n, \mathbb{R}) \to M(\mathbb{R}^{2n}, \mathbb{R})$, $W(F) = HFH^{-1}$ is an $\mathbb{R}-$ isomorphism. Moreover, the inverse of W is also an $\mathbb{R}-$isomorphism.*

Proof. It is clear from Proposition 1. □

Definition 3. *Let $\mathbb{C}^n$ ($\mathbb{R}^n$) be the complex (real) vector space.*
A transformation $A : \mathbb{C}^n \to \mathbb{C}^n$ ($A : \mathbb{R}^n \to \mathbb{R}^n$) is called unitary (orthogonal), if $< A(x), A(y) >=< x, y >$, for all $x, y \in \mathbb{C}^n$ ($\mathbb{R}^n$).

Definition 4. *A transformation $A : \mathbb{C}^n \to \mathbb{C}^n$ is called real unitary, if $< A(x), A(y) >_r =< x, y >_r$, for all $x, y \in \mathbb{C}^n$, where $<, >_r$ is the real part of $<, >$.*

Proposition 2. *A transformation $A : \mathbb{C}^n \to \mathbb{C}^n$ is real unitary if and only if the transformation $W(A)$ is orthogonal where W is the transformation in Theorem 1.*

Proof. $(\Rightarrow)$: Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a real unitary transformation. For all $x, y \in \mathbb{C}^n$, $< A(x), A(y) >_r =< x, y >_r$. Let $x = (a_1, ..., a_n) + i(b_1, ..., b_n)$ and $y = (c_1, ..., c_n) + i(d_1, ..., d_n)$, where $i = \sqrt{-1}$. Then,

$$< x, y >= a_1c_1 + b_1d_1 + ... + a_nc_n + b_nd_n + i(b_1c_1 - a_1d_1 + ... + b_nc_n - a_nd_n).$$

$$\begin{aligned} < x, y >_r &= a_1c_1 + b_1d_1 + ... + a_nc_n + b_nd_n \\ &= (Re(x), Im(x))(Re(y), Im(y))^T =< H(x), H(y) >, \end{aligned}$$

where H is the transformation defined in Proposition 1.
Since $< x, y >_r =< H(x), H(y) >$ for all $x, y \in \mathbb{C}^n$,

$$< A(x), A(y) >_r =< H(A(x)), H(A(y)) > .$$

Hence,

$$< H(A(x)), H(A(y)) >=< W(A)(H(x)), W(A)(H(y)) >=< H(x), H(y) >$$

is obtained. Consequently, the transformation $W(A)$ is orthogonal.
$(\Leftarrow)$: Let $B : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be orthogonal. Then, for all $x', y' \in \mathbb{R}^{2n}$,

$$< B(x'), B(y') >=< x', y' > .$$

Let $x' = (a_1, ..., a_n, b_1, ..., b_n)$,$y' = (c_1, ..., c_n, d_1, ..., d_n)$. By the inverse transformation of H defined in Proposition 1, $H^{-1}(x') = (a_1, ..., a_n) + i(b_1, ..., b_n)$, $H^{-1}(y') = (c_1, ..., c_n) + i(d_1, ..., d_n)$ are obtained. By the similar operations with the first part of the proof,

$$< x', y'^{-1}(x'^{-1}(y') >_r$$

and then,

$$< W^{-1}(B)(H^{-1}(x'^{-1}(B)(H^{-1}(y')) >_r =< H^{-1}(x'^{-1}(y') >_r .$$

By the above equation, the transformation $W^{-1}(B)$ is real unitary. □

Let $O(n)$ be the set of all orthogonal transformations of the $n-$ dimensional Euclidean space.

Proposition 3. *According to the operation that combines the elements, the set of all real unitary transformations is a group.*

Proof. Let A_1, A_2 be real unitary. For all $x, y \in \mathbb{C}^n$,

$$< A_1(A_2(x)), A_1(A_2(y)) >_r = < A_2(x), A_2(y) >_r = < x, y >_r .$$

Then, the composition A_1A_2 is real unitary.
Since $< I(x), I(y) >_r = < x, y >_r$, where I is identity transformation, I is a real unitary transformation.
For each real unitary transformation A, by Proposition 2, $W(A) \in O(2n)$. Then, there exists $(W(A))^{-1}$ and $(W(A))^{-1} \in O(2n)$. Since

$$(W(A))^{-1} = (HAH^{-1})^{-1} = HA^{-1}H^{-1} = W(A^{-1}),$$

A^{-1} is a real unitary transformation.
Consequently, the set of all real unitary transformations is a group. □

Denote by $U_r(n)$ the set of all real unitary transformations in the $n-$dimensional unitary space.

Definition 5. *Let V be an $n-$dimensional vector space. A transformation $F : V \to V$ is called an isometry if $\|F(x) - F(y)\| = \|x - y\|$ for all $x, y \in V$, where $\|.\|$ is a function that assigns a strictly positive length to each vector.*

Denote by $Is(\mathbb{C}^n)$ $(Is(\mathbb{R}^{2n}))$ be the set of all isometries of the unitary space $\mathbb{C}^n$ (the Euclidean space $\mathbb{R}^{2n}$).

Theorem 2. *A transformation F is an isometry on the $n-$dimensional unitary space if and only if $W(F)$ is an isometry on the $2n-$dimensional Euclidean space, where W is the transformation defined by Theorem 1.*

Proof. The proof is straightforward. □

Definition 6. *Let B be a set, G be a group and α be an action of G on B. Elements $a, b \in B$ is called $G-$equivalent if there exists $q \in G$ such that $b = \alpha(q, a)$. In this case, we write $a \overset{G}{\sim} b$.*

Example 2. *Let α be an action of $O(2)$ on $\mathbb{R}^2$, where $\alpha(g, x) = g(x)$, for all $g \in O(2)$, $x \in \mathbb{R}^2$. For $x_1 = (2, -1)$, $x_2 = (\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, there exists the transformation $g(x_1, x_2) = \frac{\sqrt{2}}{2}(x_1 - x_2, x_1 + x_2)$ such that $(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = g(2, -1)$. Then, $x_1 \overset{O(2)}{\sim} x_2$.*

Definition 7. *Let K be a set. A function $h : B \to K$ is called $G-$invariant if $a, b \in B$, $a \overset{G}{\sim} b$ implies $h(a) = h(b)$.*

Example 3. *Define the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = < x_1, x_2 >$. Since*

$$f(g(x_1), g(x_2)) = < g(x_1), g(x_2) > = < x_1, x_2 > = f(x_1, x_2)$$

for all $g \in O(2)$, the function f is $O(2)-$invariant.

Remark 2. *Let $Tr(B, K)^G$ be the set of all $G-$invariant functions $h : B \to K$.*

Definition 8. *([9], p. 11) A system $\{f_1, ..., f_m\}$, where $f_i \in Tr(B,K)^G$, will be called a complete system of $G-$invariants of the action α if $a, b \in B$,*

$$f_i(a) = f_i(b)$$

for all $i = 1, ..., m$ imply $a \overset{G}{\sim} b$.

Remark 3. *Let $P = \{f_1, ..., f_m\} \subset Tr(B,K)^G$. Denote by $Tr(B,K;P)$ the set of all $h : B \to K$ such that h is a function of the system P.*

Proposition 4. *Let $P = \{f_1, ..., f_m\}$ be a complete system of $G-$invariant functions on B. Then $Tr(B,K)^G = Tr(B,K;P)$.*

Proof. Proof is given ([9], p. 11, Theorem 1.1) □

Definition 9. *([9], p. 11) A complete system $P = \{f_1, ..., f_m\}$ of $G-$invariant functions will be called a minimal complete system if $P \setminus \{f_i\}$ is not complete for any $i = 1, ..., m$.*

Proposition 5. *Let $P = \{f_1, ..., f_m\}$ be a complete system, where $f_i \in Tr(B,K)^G$. Then P is a minimal complete system iff $f_j \notin Tr(B,K;P \setminus \{f_j\})$ for all $j = 1, ..., m$.*

Proof. Proof is given ([2], p. 545, Proposition 2) □

3. Complete System of Invariants

Proposition 6. *Let $\{x_1, ..., x_m\}$, $\{y_1, ..., y_m\} \subset \mathbb{C}^n$. Then,*
$\{x_1, ..., x_m\} \overset{Is(\mathbb{C}^n)}{\sim} \{y_1, ..., y_m\}$ if and only if
$\{x_1 - x_m, ..., x_{m-1} - x_m\} \overset{U_r(n)}{\sim} \{y_1 - y_m, ..., y_{m-1} - y_m\}$.

Proof. $(\Rightarrow)$: Let $\{x_1, ..., x_m\} \overset{Is(\mathbb{C}^n)}{\sim} \{y_1, ..., y_m\}$. Then, there exists an isometry $F \in Is(\mathbb{C}^n)$ such that $y_i = F(x_i)$, $i = 1, ..., m$. By the transformation H in Proposition 1, we get $H(y_i) = H(F(x_i))$, $i = 1, ..., m$. Therefore,

$$\begin{aligned} H(y_i) =& H(F(H^{-1}H)(x_i)), && i = 1, ..., m \\ H(y_i) =& HFH^{-1}(H(x_i)), && i = 1, ..., m. \end{aligned}$$

By Theorem 2,

$$HFH^{-1} = W(F) \in Is(\mathbb{R}^{2n}).$$

Thus, for the systems $\{H(x_1), ..., H(x_m)\}$, $\{H(y_1), ..., H(y_m)\} \subset \mathbb{R}^{2n}$, there exists an isometry $W(F) \in Is(\mathbb{R}^{2n})$ such that

$$H(y_i) = W(F)(H(x_i)), i = 1, ..., m.$$

Then,

$$H(x_1), ..., H(x_m)\} \overset{Is(\mathbb{R}^{2n})}{\sim} \{H(y_1), ..., H(y_m)\}$$

is obtained. By Theorem in ([3], p. 3, Theorem 1), if
$H(x_1),...,H(x_m) \overset{Is(\mathbb{R}^{2n})}{\sim} \{H(y_1),...,H(y_m)\}$, then

$$\{H(x_1)-H(x_m),...,H(x_{m-1})-H(x_m)\} \overset{O(2n)}{\sim} \{H(y_1)-H(y_m),...,H(y_{m-1})-H(x_m)\}.$$

Since the transformation H in Proposition 1 is $\mathbb{R}-$linear,

$$\{H(x_1-x_m),...,H(x_{m-1}-x_m)\} \overset{O(2n)}{\sim} \{H(y_1-y_m),...,H(y_{m-1}-x_m)\}.$$

Then, there exists an element $A' \in O(2n)$ such that for $i=1,...,m-1$

$$H(y_i-y_m) = A'(H(x_i-x_m)).$$

If we multiply both sides of the equation by the map H^{-1}, we get

$$\begin{aligned} H^{-1}(H(y_i-y_m)) =& H^{-1}(A'(H(x_i-x_m))), && i=1,...,m-1 \\ y_i-y_m =& H^{-1}(A'^{-1}H)(H(x_i-x_m))), && i=1,...,m-1 \\ y_i-y_m =& (H^{-1}A'H)(x_i-x_m), && i=1,...,m-1. \end{aligned}$$

Let us put $H^{-1}A'H = A$. Then, $A'^{-1} = W(A)$. By Proposition 2, $A \in U_r(n)$. Since $A \in U_r(n)$ and $y_i - y_m = A(x_i - x_m)$, $i=1,...,m-1$, we get $\{x_1-x_m,...,x_{m-1}-x_m\} \overset{U_r(n)}{\sim} \{y_1-y_m,...,y_{m-1}-y_m\}$.

($\Leftarrow$) : Let $\{x_1-x_m,...,x_{m-1}-x_m\} \overset{U_r(n)}{\sim} \{y_1-y_m,...,y_{m-1}-y_m\}$. Then, there exists an element $A \in U_r(n)$ such that $y_i - y_m = A(x_i - x_m)$, $i=1,...,m-1$. The transformation H in Proposition 1, we get $H(y_i-y_m) = H(A(x_i-x_m))$, $i=1,...,m-1$. Therefore,

$$\begin{aligned} H(y_i-y_m) =& H(A(H^{-1}H)(x_i-x_m)) && i=1,...,m-1 \\ H(y_i-y_m) =& (HAH^{-1})(H(x_i-x_m)) && i=1,...,m-1. \end{aligned}$$

By Proposition 2, $HAH^{-1} = W(A) \in O(2n)$. Since

$$H(y_i-y_m) = (W(A))(H(x_i-x_m)), \qquad i=1,...,m-1,$$

for the systems

$$\{H(x_1-x_m),...,H(x_{m-1}-x_m)\}, \{H(y_1-y_m),...,H(y_{m-1}-y_m)\} \subset \mathbb{R}^{2n},$$

$$\{H(x_1-x_m),...,H(x_{m-1}-x_m)\} \overset{O(2n)}{\sim} \{H(y_1-y_m),...,H(y_{m-1}-y_m)\}$$

is obtained. By Proposition 1,

$$\{H(x_1)-H(x_m),...,H(x_{m-1})-H(x_m)\} \overset{O(2n)}{\sim} \{H(y_1)-H(y_m),...,H(y_{m-1})-H(y_m)\}.$$

By Theorem in ([3], p. 3, Theorem 1),

$$\{H(x_1),...,H(x_m)\} \overset{Is(\mathbb{R}^{2n})}{\sim} \{H(y_1),...,H(y_m)\}.$$

Then, there exists $F' \in Is(\mathbb{R}^{2n})$ such that

$$H(y_i) = F'(H(x_i)), \qquad i = 1, ..., m.$$

If we multiply both sides of the equation by the map H^{-1}, we get

$$\begin{aligned} H^{-1}(H(y_i)) =& H^{-1}(F'(H(x_i))), && i = 1, ..., m \\ y_i =& H^{-1}(F'^{-1}H)(H(x_i))), && i = 1, ..., m \\ y_i =& (H^{-1}F'H)(x_i), && i = 1, ..., m. \end{aligned}$$

Let us put $H^{-1}F'H = F$. Then, $F'^{-1} = W(F)$.
By Theorem 2, $F \in Is(\mathbb{C}^n)$. Since $F \in Is(\mathbb{C}^n)$ and, $y_i = F(x_i)$, $i = 1, ..., m$, $\{x_1, ..., x_m\} \overset{Is(\mathbb{C}^n)}{\sim} \{y_1, ..., y_m\}$ is obtained. □

Proposition 7. *Let* $\{x_1, ..., x_m\}$, $\{y_1, ..., y_m\} \subset \mathbb{C}^n$. *Then,* $\{x_1, ..., x_m\} \overset{U_r(n)}{\sim} \{y_1, ..., y_m\}$ *if and only if* $< x_i, x_j >_r = < y_i, y_j >_r$, $1 \leq i \leq j \leq m$.

Proof. $(\Rightarrow)$: For the systems $\{x_1, ..., x_m\}$, $\{y_1, ..., y_m\} \subset \mathbb{C}^n$, let

$$\{x_1, ..., x_m\} \overset{U_r(n)}{\sim} \{y_1, ..., y_m\}.$$

Then, there exists an element $A \in U_r(n)$ such that $y_i = A(x_i)$, $i = 1, ..., m$. Therefore, we get

$$< x_i, x_j >_r = < A(x_i), A(x_j) >_r = < y_i, y_j >_r, \qquad 1 \leq i \leq j \leq m.$$

$(\Leftarrow)$: For the systems $\{x_1, ..., x_m\}$, $\{y_1, ..., y_m\} \subset \mathbb{C}^n$, let

$$< x_i, x_j >_r = < y_i, y_j >_r, \qquad 1 \leq i \leq j \leq m.$$

Since $< x_i, x_j > = \overline{< x_j, x_i >}$, $1 \leq i \leq j \leq m$, we get $< x_i, x_j >_r = < y_i, y_j >_r$, $1 \leq i, j \leq m$.
For all $z_1, z_2 \in \mathbb{C}^n$, let $z_1 = (a_1, ..., a_n) + i(b_1, ..., b_n)$, $z_2 = (c_1, ..., c_n) + i(d_1, ..., d_n)$. Then,

$$< z_1, z_2 >_r = a_1c_1 + ... + a_nc_n + b_1d_1 + ... + b_nd_n.$$

By Proposition 1, $H(z_1) = (Re(z_1), Im(z_1))$, $H(z_2) = (Re(z_2), Im(z_2))$. Therefore,

$$< H(z_1), H(z_2) > = a_1c_1 + ... + a_nc_n + b_1d_1 + ... + b_nd_n.$$

and we get

$$< z_1, z_2 >_r = < H(z_1), H(z_2) > .$$

By using this equation, we have

$$< x_i, x_j >_r = < H(x_i), H(x_j) >, \qquad 1 \leq i, j \leq m.$$

Since $< x_i, x_j >_r = < y_i, y_j >_r$, $1 \leq i, j \leq m$, it is obtained that

$$< H(x_i), H(x_j) > = < H(y_i), H(y_j) >, \qquad 1 \leq i, j \leq m.$$

By Theorem ([3], p. 4, Theorem 3),

$$\{H(x_1), ..., H(x_m)\} \overset{O(2n))}{\sim} \{H(y_1), ..., H(y_m)\}.$$

So, there exists a transformation $A' \in O(2n)$ such that $H(y_i) = A'(H(x_i))$. If we multiply both sides of the equation by the map H^{-1}, we get

$$\begin{aligned} H^{-1}(H(y_i)) =& H^{-1}(A'(H(x_i))), && i = 1, ..., m \\ y_i =& H^{-1}(A'^{-1})(H(x_i))), && i = 1, ..., m \\ y_i =& (H^{-1}A'H)(x_i), && i = 1, ..., m. \end{aligned}$$

Let us put $H^{-1}A'H = A$. Then, $A'^{-1} = W(A)$. By Proposition 2, $A \in U_r(n)$. Since $A \in U_r(n)$ and $y_i = A(x_i)$, $i = 1, ..., m$, we get $\{x_1, ..., x_m\} \overset{U_r(n)}{\sim} \{y_1, ..., y_m\}$. □

Corollary 1. *Let* $\{x_1, ..., x_m\}$, $\{y_1, ..., y_m\} \subset \mathbb{C}^n$. *Then,*

$$\{x_1, ..., x_m\} \overset{Is(\mathbb{C}^n)}{\sim} \{y_1, ..., y_m\}$$

if and only if

$$< x_i - x_m, x_j - x_m >_r = < y_i - y_m, y_j - y_m >_r, \qquad 1 \leq i \leq j \leq m - 1.$$

Proof. By Proposition 6, for the systems $\{x_1, ..., x_m\}$, $\{y_1, ..., y_m\} \subset \mathbb{C}^n$,

$$\{x_1, ..., x_m\} \overset{Is(\mathbb{C}^n)}{\sim} \{y_1, ..., y_m\}$$

if and only if

$$\{x_1 - x_m, ..., x_{m-1} - x_m\} \overset{U_r(n)}{\sim} \{y_1 - y_m, ..., y_{m-1} - y_m\}.$$

By Proposition 7,

$$\{x_1 - x_m, ..., x_{m-1} - x_m\} \overset{U_r(n)}{\sim} \{y_1 - y_m, ..., y_{m-1} - y_m\}$$

if and only if

$$< x_i - x_m, x_j - x_m >_r = < y_i - y_m, y_j - y_m >_r, \qquad 1 \leq i \leq j \leq m - 1.$$

□

Theorem 3. *The system* $IsT = \{< x_j - x_m, x_k - x_m >_r | \ 1 \leq j \leq k \leq m - 1\}$ *is a minimal complete system of* $Is(\mathbb{C}^n)$ *invariants of the vectors* $x_1, ..., x_m \in \mathbb{C}^n$.

Proof. Any subset of IsT is contained in a subset of IsT_1, where
$IsT_1 = IsT - < x_p - x_m, x_q - x_m >_r$ for all $1 \leq p \leq q \leq m - 1$. Now, let us show that IsT_1 is not a complete system of $Is(\mathbb{C}^n)$ invariants. Firstly, assume that $p \neq q$. Consider the systems $\{x_1, ..., x_m\}$, $\{y_1, ..., y_m\} \subset \mathbb{C}^n$. For $1 \leq j \leq m - 1$ and $i = \sqrt{-1}$, $x_p = x_q = (e^{i\theta}, 0, ..., 0)$, $x_m = (0, 0, e^{i\theta}, 0, ..., 0)$, $x_j = (0, ..., 0)$, $j \neq p$, $j \neq q$. Then,

$$< x_j - x_m, x_k - x_m >_r = < y_j - y_m, y_k - y_m >_r = 1,$$

where $i \leq j \leq k \leq m-1$ and $(j,k) \neq (p,q)$.
However, these equations do not mean $\{x_1,...,x_m\} \overset{Is(\mathbb{C}^n)}{\sim} \{y_1,...,y_m\}$. Since

$$2 =< x_p - x_m, x_q - x_m >_r \neq < y_p - y_m, y_q - y_m >_r = 1,$$

the systems $\{x_1,...,x_m\}$, $\{y_1,...,y_m\}$ are not $Is(\mathbb{C}^n)$ equivalent. This also shows that IsT_1 is not a complete system of $Is(\mathbb{C}^n)$ invariants. Secondly, assume that $p = q$. Consider the systems $\{x_1,...,x_m\}$, $\{y_1,...,y_m\}$. For $1 \leq j \leq m-1$ and $i = \sqrt{-1}$, $x_p = (e^{i\theta}, 0, ..., 0)$, $x_m = (0, 0, e^{i\theta}, 0, ..., 0)$, $x_j = (0, ..., 0)$, $j \neq p$ and $y_p = (e^{i\theta}, 1, 0, ..., 0)$, $y_m = (0, 0, e^{i\theta}, 0, ..., 0)$, $y_j =: (0, ..., 0)$, $j \neq p$. Then,

$$< x_j - x_m, x_k - x_m >_r =< y_j - y_m, y_k - y_m >_r = 1,$$

where $1 \leq j \leq k \leq m-1$ and $(j,k) \neq (p,p)$.
However, these equations do not mean $\{x_1,...,x_m\} \overset{Is(\mathbb{C}^n)}{\sim} \{y_1,...,y_m\}$. Since

$$2 =< x_p - x_m, x_q - x_m >_r \neq < y_p - y_m, y_q - y_m >_r = 3,$$

the systems $\{x_1,...,x_m\}$, $\{y_1,...,y_m\}$ are not $Is(\mathbb{C}^n)$ equivalent. This also shows that IsT_1 is not a complete system of $Is(\mathbb{C}^n)$ invariants. $\square$

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