

Relativistic composite systems and minimal coupling

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The problem of minimal coupling of relativistic composite systems having internal degrees of freedom is investigated. The ambiguity in the introduction of various internal coordinates with the same mass spectrum can be solved by probing the system with external electromagnetic interactions. For the generalized Majorana equation this procedure shows that the charged constituent of the system performs a two-dimensional Kepler motion in the Klein-Gordon approximation and not an oscillator motion.

I. INTRODUCTION

We are concerned with the problem of determining the internal structure or the internal coordinates of a quantum system from the knowledge of its global quantum numbers and mass spectrum. This is one of the important theoretical problems in hadron physics where one is trying to infer the internal dynamics of hadrons from their multiplet structure, mass levels, quantum numbers, form factors, etc. A question often raised, for example, in the naive quark model is whether and how one can use nonrelativistic internal coordinates in a relativistic treatment of the hadrons as a whole.¹ In more mathematical terms the problem amounts to a joint treatment of the generators of the Poincaré group with operators describing the internal quantum numbers of the system, an area of considerable theoretical activity.

We shall concentrate on certain infinite-component wave equations describing in a covariant way internal structure of composite systems. In a previous paper we have discussed the introduction of internal coordinates into the infinite-component Majorana wave equation,² which is related to Dirac's new wave equation without negative-energy solutions.³ It was shown that both a two-dimensional oscillator or a two-dimensional Coulomb-type nonrelativistic (or relativistic) internal dynamics could be introduced, and the relation between these two types of relative coordinates was given.⁴ There is therefore an ambiguity in determining the internal dynamics from the global relativistic wave equation, or from the mass spectrum alone.

Thus different types of internal dynamics are compatible with the same mass spectrum. We can, however, determine the internal dynamics more closely by probing with external interactions. This is indeed how one would and does proceed experimentally. Accordingly we couple the Majorana equation minimally to an external electromagnetic field, and analyze the motion of the charged point-like constituent of the system. The correct inter-

nal dynamics can be selected by determining in which of the internal coordinates the matrix elements contain the correct coupling factors $\vec{p}_1 \cdot \vec{A}(\vec{x}_1)$ or $\exp(i\vec{q} \cdot \vec{x}_1)$, where \vec{x}_1 and \vec{p}_1 are the coordinates of a charged constituent and \vec{q} is the momentum of the photon. We carry out this analysis and find, somewhat surprisingly, that the charged constituent performs a Kepler motion as a result of the minimal coupling of the Majorana equation, and not an oscillatory motion. The oscillator coordinates were used by Dirac³ and recently by others^{5,6} and the difficulty with the minimal coupling was encountered. Our result therefore solves this problem: Probing of the Majorana system by an external electromagnetic field via minimal coupling reveals long-range forces of Coulomb type and a two-dimensional internal structure between two constituents, one of them being charged.

Our result can also be derived by reduction from a more general $SO(4, 2)$ infinite-component wave equation.⁷ The latter describes, when coupled minimally to an electromagnetic field, a three-dimensional internal Coulomb motion. If we delete the z coordinate, for example, and have a two-dimensional internal motion, both the dynamical group and the wave equation reduce exactly to the generalized Majorana equation that we have considered. The dynamical group underlying the Majorana equation, $SO(3, 1)$, is not large enough to support a three-dimensional internal motion; there are not enough quantum numbers. The $SO(4, 2)$ wave equation contains, in addition to spin, the principal quantum number n and therefore can be realized by a three-dimensional internal motion.

Instead of the wave equation one can use pure algebraic relations in embedding the Poincaré generators and internal quantum numbers.⁸ But these two approaches are identical.⁹ The wave-equation approach is much more compact and simpler, and more convenient from the point of view of minimal coupling. There is considerable literature in the purely algebraic treatment of the Poincaré Lie algebra and internal quantum numbers.^{10,11}

II. INTERNAL COORDINATES OF THE MAJORANA EQUATION

The generalized Majorana equation in momentum space is

$$(J_\mu P^\mu - K)\psi(\mathbf{p}) = 0, \quad (2.1)$$

where the current operator is given by

$$J_\mu = \Gamma_\mu - \lambda P_\mu. \quad (2.1')$$

Here the four-vector operators Γ_μ together with the generators \vec{J} and \vec{M} of the Lorentz group $SO(3, 1)$ form the Lie algebra of $SO(3, 2)$. The states of the system with timelike momenta are either realized in a discrete basis $|j, m\rangle$ with all spins $j=0, 1, 2, \dots$, or in a continuous basis in terms of the coordinates (x, y) , or the coordinates (r, θ) . The former we call the harmonic-oscillator coordinates, the latter the Kepler coordinates. The generators of $SO(3, 2)$ in the harmonic-oscillator coordinates are realized by^{12,4}

$$\begin{aligned} L_{23} &\equiv L_1 = \frac{1}{4}i(z^2 - \bar{z}^2 + \partial^2 - \bar{\partial}^2), \\ L_{31} &\equiv L_2 = \frac{1}{4}(z^2 + \bar{z}^2 - \partial^2 - \bar{\partial}^2), \\ L_{12} &\equiv L_3 = \frac{1}{2}(z\partial - \bar{z}\bar{\partial}), \\ L_{10} &\equiv M_1 = \frac{1}{4}(z^2 + \bar{z}^2 + \partial^2 + \bar{\partial}^2), \\ L_{20} &\equiv M_2 = -\frac{1}{4}i(z^2 - \bar{z}^2 - \partial^2 + \bar{\partial}^2), \\ L_{30} &\equiv M_3 = -\frac{1}{2}i(z\partial + \bar{z}\bar{\partial} - 2), \\ L_{15} &\equiv \Gamma_1 = -\frac{1}{2}i(z\bar{\partial} + \bar{z}\partial), \\ L_{25} &\equiv \Gamma_2 = (-\frac{1}{2})(z\bar{\partial} - \bar{z}\partial), \\ L_{35} &\equiv \Gamma_3 = -\frac{1}{2}(z\bar{z} + \partial\bar{\partial}), \\ L_{05} &\equiv \Gamma_0 = \frac{1}{2}(z\bar{z} - \partial\bar{\partial}), \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} z &= (1/s_0)(x + iy), \\ \partial &= \frac{1}{2}s_0(\partial/\partial x - i\partial/\partial y), \end{aligned} \quad (2.3)$$

with s_0 being an arbitrary constant of dimension of a length so that z is dimensionless. In (2.2) the commutation relations of the generators of $SO(3, 2)$ are given by

$$\begin{aligned} [L_{ab}, L_{cd}] &= -i(g_{ac}L_{bd} + L_{ac}g_{bd} - g_{ad}L_{bc} \\ &\quad - g_{bc}L_{ad}), \end{aligned}$$

where the metric is $(+---)$ and $a, b = 0, 1, 2, 3, 5$. Physically, $L_{ij} = \epsilon_{ijk}L_k$ are the angular momentum operators, L_{i0} are the generators of pure Lorentz transformations (boosts), and $\Gamma_\mu = (L_{05}, L_{i5})$ transforms as a four-vector with respect to the Lorentz group. The realization (2.2) is related to that of Staunton *et al.*^{5,8} in a simple manner.⁹

The realization in terms of the Kepler coordinates is given by

$$\begin{aligned} L_1 &= -\frac{1}{4}\left[\frac{r}{r_0}\sin\theta - 4r_0\sin\theta\left(r\partial_r^2 - \frac{1}{r}\partial_\theta^2\right) - 4r_0\cos\theta\left(2\partial_r\partial_\theta - \frac{1}{r}\partial_\theta\right)\right], \\ L_2 &= \frac{1}{4}\left[\frac{r}{r_0}\cos\theta - 4r_0\cos\theta\left(r\partial_r^2 - \frac{1}{r}\partial_\theta^2\right) + 4r_0\sin\theta\left(2\partial_r\partial_\theta - \frac{1}{r}\partial_\theta\right)\right], \\ L_3 &= -i\partial_\theta; \\ M_1 &= \frac{1}{4}\left[\frac{r}{r_0}\cos\theta + 4r_0\cos\theta\left(r\partial_r^2 - \frac{1}{r}\partial_\theta^2\right) - 4r_0\sin\theta\left(2\partial_r\partial_\theta - \frac{1}{r}\partial_\theta\right)\right], \\ M_2 &= \frac{1}{4}\left[\frac{r}{r_0}\sin\theta + 4r_0\sin\theta\left(r\partial_r^2 - \frac{1}{r}\partial_\theta^2\right) + 4r_0\cos\theta\left(2\partial_r\partial_\theta - \frac{1}{r}\partial_\theta\right)\right], \\ M_3 &= -i(r\partial_r - 1); \\ \Gamma_1 &= -i(r\cos\theta\partial_r - \sin\theta\partial_\theta), \\ \Gamma_2 &= -i(r\sin\theta\partial_r + \cos\theta\partial_\theta), \\ \Gamma_3 &= -rr_0\nabla^2 - r/4r_0, \\ \Gamma_0 &= -rr_0\nabla^2 + r/4r_0. \end{aligned} \quad (2.4)$$

This realization is obtained from that of Eq. (2.2) by the following transformation of coordinates:

$$z = (r/2r_0)^{1/2}e^{i\theta/2}. \quad (2.5)$$

For timelike momenta and in the rest frame of the system Eq. (2.1) becomes

$$(\Gamma_0 M - \lambda M^2 - K)\psi(0) = 0, \quad (2.6)$$

where M is the total mass of the system ($P_0 = M$). Inserting for Γ_0 the expression either from (2.2) or from (2.4) we recognize immediately the Schrödinger equation for a particle with relative internal coordinates (x, y) moving in an oscillator potential, or with relative coordinates (r, θ) moving in a two-dimensional Coulomb potential.⁴

Thus, nonrelativistic internal coordinates are perfectly compatible with the relativistic motion of the composite system in the sense of Eq. (2.6). The question now is whether these internal coordinates are in any sense also physically meaningful; i.e., do they correspond to actual constituents?

III. MINIMAL COUPLING OF A NONRELATIVISTIC TWO-BODY SYSTEM

Let the masses, positions, and momenta of the particles be m_i , \vec{r}_i , and \vec{p}_i , $i = 1, 2$, respectively. The relative coordinates and momenta are, as usual,

$$\begin{aligned}\bar{\mathbf{r}} &= \bar{\mathbf{r}}_1 - \bar{\mathbf{r}}_2, \quad \bar{\mathbf{R}} = (m_1/M)\bar{\mathbf{r}}_1 + (m_2/M)\bar{\mathbf{r}}_2, \\ \bar{\mathbf{p}} &= (m_2/M)\bar{\mathbf{p}}_1 - (m_1/M)\bar{\mathbf{p}}_2, \quad \bar{\mathbf{P}} = \bar{\mathbf{p}}_1 + \bar{\mathbf{p}}_2,\end{aligned}\quad (3.1)$$

where $M = m_1 + m_2$, and the reduced mass is $\mu = m_1 m_2 / M$.

Let the particle 1 be charged and we couple the system minimally to the electromagnetic field. The interaction is

$$-\frac{e_1}{2m_1} [\bar{\mathbf{p}}_1 \cdot \bar{\mathbf{A}}(\bar{\mathbf{r}}_1) + \bar{\mathbf{A}}(\bar{\mathbf{r}}_1) \cdot \bar{\mathbf{p}}_1]$$

and the corresponding matrix element between initial and final states of the total system is

$$M = \int d\bar{\mathbf{r}}_1 d\bar{\mathbf{r}}_2 \Psi_f^* \left\{ -\frac{e_1}{2m_1} [\bar{\mathbf{p}}_1 \cdot \bar{\mathbf{A}}(\bar{\mathbf{r}}_1) + \bar{\mathbf{A}}(\bar{\mathbf{r}}_1) \cdot \bar{\mathbf{p}}_1] \right\} \Psi_i. \quad (3.2)$$

The total wave function of the system has the form

$$\Psi(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2) = \phi(\bar{\mathbf{R}}) \psi(\bar{\mathbf{r}}), \quad (3.3)$$

where $\phi(\bar{\mathbf{R}}) = e^{i\bar{\mathbf{p}} \cdot \bar{\mathbf{R}}}$ represent the plane wave of the center of mass and $\psi(r)$ is the wave function of the relative internal motion. We take the external field to be a plane wave

$$\bar{\mathbf{A}}(\bar{\mathbf{r}}_1) = (2\pi/qV)^{1/2} \bar{\boldsymbol{\epsilon}} e^{i\bar{\mathbf{q}} \cdot \bar{\mathbf{r}}_1}, \quad (3.4)$$

where $\bar{\mathbf{q}}$ and $\bar{\boldsymbol{\epsilon}}$ characterize the momentum and polarization of the incoming photon. In terms of the relative coordinates the matrix element (3.3) becomes

$$\begin{aligned}M &\propto \bar{\boldsymbol{\epsilon}} \cdot \int d\bar{\mathbf{R}} d\bar{\mathbf{r}} \exp[i\bar{\mathbf{q}} \cdot (\bar{\mathbf{R}} - \bar{\mathbf{P}}_j + \bar{\mathbf{P}}_i) \cdot \bar{\mathbf{R}}] \\ &\quad \times \Psi_f^*(\bar{\mathbf{r}}) \left(2\bar{\mathbf{p}} + \frac{m_1 - m_2}{M} \bar{\mathbf{q}} + \frac{2m_1}{M} \bar{\mathbf{p}}_i \right) \\ &\quad \times \exp\left(i\bar{\mathbf{q}} \cdot \frac{m_2}{M} \bar{\mathbf{r}}\right) \psi_i(\bar{\mathbf{r}}),\end{aligned}\quad (3.5)$$

or, after the R integration

$$\begin{aligned}M &\propto \bar{\boldsymbol{\epsilon}} \cdot \int d\bar{\mathbf{r}} \psi_f^*(\bar{\mathbf{r}}) \exp\left(i\bar{\mathbf{P}}_f \cdot \frac{m_2}{M} \bar{\mathbf{r}}\right) \left[\bar{\mathbf{p}} + \frac{1}{2}(\bar{\mathbf{P}}_f + \bar{\mathbf{P}}_i)\right] \\ &\quad \times \exp\left(-i\bar{\mathbf{P}}_i \cdot \frac{m_2}{M} \bar{\mathbf{r}}\right) \psi_i(\bar{\mathbf{r}}).\end{aligned}\quad (3.6)$$

We interpret

$$\bar{\mathbf{J}} = \bar{\mathbf{p}} + \frac{1}{2}(\bar{\mathbf{P}}_f + \bar{\mathbf{P}}_i) \quad (3.7)$$

as the current operator, while the boost is

$$\exp[-i\bar{\mathbf{P}} \cdot (m_2/M)\bar{\mathbf{r}}]. \quad (3.8)$$

Thus the matrix element (3.6) can be interpreted as that of the current $\bar{\mathbf{J}}$ between two boosted rest-frame wave functions. The formulas of this chapter up to this point are quite general and valid in any dimension.⁷ We shall now specialize to the two-dimensional internal motion. In this case

all the 3-vectors occurring in Eqs. (3.1)–(3.8) are 2-vectors. Only the x and y components of the external photon momentum $\bar{\mathbf{q}}$ contribute to the internal excitation, while the z component of $\bar{\mathbf{q}}$ boosts the center of mass of the system.

From the realization (2.4) in Kepler coordinates we see that if the internal dynamics are of the Coulomb type we can express the current (3.7) and the generator $(m_2/M)\bar{\mathbf{r}}$ of the boost (3.8) directly in terms of the $SO(3, 2)$ -group generators as follows:

$$\begin{aligned}\bar{\mathbf{p}} &= -i \left(\cos\theta \partial_r - \frac{1}{r} \sin\theta \partial_\theta, \sin\theta \partial_r + \frac{1}{r} \cos\theta \partial_\theta \right) \\ &= \frac{1}{r} (\Gamma_1, \Gamma_2) \\ &= \frac{1}{r} \bar{\Gamma},\end{aligned}\quad (3.9)$$

$$r = 2r_0(\Gamma_0 - \Gamma_3),$$

$$\bar{\mathbf{r}} = 2r_0(\bar{\mathbf{M}} + C\bar{\mathbf{L}}), \quad \bar{\mathbf{M}} \equiv (M_1, M_2), \quad \bar{\mathbf{L}} \equiv (L_1, L_2)$$

where

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence the current operator becomes

$$\bar{\mathbf{J}} = (1/r) [\bar{\Gamma} + 2r_0(\Gamma_0 - \Gamma_3)^{1/2} (\bar{\mathbf{P}}_f + \bar{\mathbf{P}}_i)]. \quad (3.10)$$

The factor $(1/r)$ cancels with the factor r in the two-dimensional volume element $d\bar{\mathbf{r}} = r dr d\theta$, and we can write the matrix element entirely as a matrix element over the group states. Note that the group metric is $(1/2\pi) dz d\bar{z} = (1/8r_0\pi) dr d\theta$. The result is

$$\begin{aligned}M &\propto \bar{\boldsymbol{\epsilon}} \cdot \int dr d\theta \psi_f^*(r) \exp\left[i\bar{\mathbf{P}}_f \cdot \frac{m_2}{M} 2r_0(\bar{\mathbf{M}} + C\bar{\mathbf{L}})\right] \\ &\quad \times [\bar{\Gamma} + 2r_0(\Gamma_0 - \Gamma_3)^{1/2} (\bar{\mathbf{P}}_f + \bar{\mathbf{P}}_i)] \\ &\quad \times \exp\left[-i\bar{\mathbf{P}}_i \cdot \frac{m_2}{M} 2r_0(\bar{\mathbf{M}} + C\bar{\mathbf{L}})\right] \psi_i(r).\end{aligned}\quad (3.11)$$

Here $\bar{\Gamma}$, Γ_0 , Γ_3 , and $(\bar{\mathbf{M}} + C\bar{\mathbf{L}})$ are operators acting on the states $|f\rangle$ and $|i\rangle$, the rest-frame states of the system, and $\bar{\mathbf{P}}_f$, $\bar{\mathbf{P}}_i$ are boosting parameters. Equation (3.11) can be written in a more compact and suggestive way as

$$M \propto \bar{\boldsymbol{\epsilon}} \cdot \langle f, \bar{\mathbf{P}}_f | [\bar{\Gamma} + 2r_0(\Gamma_0 - \Gamma_3)^{1/2} (\bar{\mathbf{P}}_f + \bar{\mathbf{P}}_i)] | i, \bar{\mathbf{P}}_i \rangle.$$

If the internal coordinates were the harmonic-oscillator coordinates it would not be possible to write the matrix element of interaction as in (3.11), linear in the group generators.

In the next section we shall show that Eq. (3.11) is precisely the vertex function of a conserved current operator (i.e., form factor) obtained from

an infinite-component wave equation. At the same time we shall pass to the relativistic kinematics so that Eq. (3.11) will be the nonrelativistic limit of the form factors. Finally, we shall pass from the covariant equation in two dimensions to the generalized Majorana equation.

IV. COVARIANT WAVE EQUATION FOR TWO-DIMENSIONAL INTERNAL MOTION

In this section we shall transform the generalized Majorana equation (2.1), or

$$(\Gamma_\mu P^\mu - \lambda P_\mu P^\mu - K)\psi(p) = 0, \quad (4.1)$$

into a form in which the *two-dimensional* internal dynamics is explicitly exhibited in a "covariant" way, that is, covariant with respect to the Lorentz transformations in 2+1 dimensions. The transformed equation is

$$[(\alpha_1 \Gamma_A + \alpha_2 P_A + \alpha_3 \Gamma_3 P_A) P^A + \beta \Gamma_3 + \gamma] \phi_2(\vec{p}) = 0, \quad (4.2)$$

with

$$\alpha_1 = 1, \quad \alpha_2 = -\alpha/2m_2, \quad \alpha_3 = 1/2m_2, \\ \beta = \frac{m_2^2 - m_1^2}{2m_2}, \quad \gamma = \alpha \frac{m_1^2 + m_2^2}{2m_2}.$$

Here $A, B = 0, 1, 2$; Γ_3 is a scalar with respect to the 2+1 Lorentz group. Hence the equation is covariant relative to this group.

We note that Eq. (4.2) has exactly the same form as the $SO(4, 2)$ equation describing the three-dimensional internal dynamics and covariant with respect to the usual Lorentz group.⁷ The only difference is that the z coordinate is missing in the internal space, hence the corresponding group generators.

We can either transform (4.1) into (4.2) or vice versa by the inverse transformation. We shall go in the second direction. By a two-dimensional Lorentz transformation with generators $\vec{M}_{(2)} = (M_1, M_2)$ we go to the rest frame of Eq. (4.2):

$$\phi_2(\vec{p}_{(2)}) = \exp[i\vec{\xi}_{(2)}(\vec{p}_{(2)}) \cdot \vec{M}_{(2)}] \phi_2(0), \quad (4.3)$$

where the rapidity of the transformation is given by

$$\vec{\xi}_{(2)}(\vec{p}_{(2)}) = (\vec{p}_{(2)}/|\vec{p}_{(2)}|) \sinh^{-1}[\vec{p}_{(2)}/M(2)].$$

Here $M(2)$ is the two-dimensional relativistic mass

$$M(2)^2 = P_A P^A = P_0^2 - (P_1^2 + P_2^2).$$

Equation (4.2) in the rest frame is then

$$\left(M(2)\Gamma_0 + \frac{1}{2m_2} [M(2)^2 + m_2^2 - m_1^2] \Gamma_3 \right. \\ \left. - \frac{\alpha}{2m_2} [M(2)^2 - m_1^2 - m_2^2] \right) \phi_2(0) = 0. \quad (4.4)$$

Note that in this section two-dimensional vectors are denoted by $\vec{p}_{(2)}$.

The relative coefficients of Γ_0 and Γ_3 in (4.4) can be changed by a hyperbolic rotation in the 03 plane which is the dilatation operation for the 2+1 space and is generated by M_3 . In general the action of such a dilatation is

$$e^{-i\delta M_3} \Gamma_0 e^{i\delta M_3} = \cosh \delta \Gamma_0 + \sinh \delta \Gamma_3, \quad (4.5)$$

$$e^{-i\delta M_3} \Gamma_3 e^{i\delta M_3} = \sinh \delta \Gamma_0 + \cosh \delta \Gamma_3$$

or, using the Kepler realization (2.4), i.e.,

$$\Gamma_0 - \Gamma_3 = r/2r_0, \quad \Gamma_0 + \Gamma_3 = -2r r_0 \nabla^2$$

we have

$$e^{-i\delta M_3} (r \nabla^2) e^{i\delta M_3} = d r \nabla^2 \\ e^{-i\delta M_3} r e^{-i\delta M_3} = r/d, \quad (4.5')$$

where the parameter of the dilatation is given by

$$\tanh \delta = (d^2 - 1)/(d^2 + 1),$$

i.e., we have a scale transformation scaling r by a factor $1/d$, and $r \nabla^2$ by a factor d .

For convenience we perform two such successive dilatations (which is of course equivalent to a single dilatation). First we put

$$\phi_2(0) = \exp(-i\eta M_3) \tilde{\phi}_2(0), \quad (4.6)$$

with

$$\tanh \eta = \frac{b_2 c_1 - b_1 c_2}{b_2 c_2 - b_1 c_1}, \quad (4.7)$$

where

$$b_1 = M(2) - (E_1^2 - m_1^2)/M(2), \\ b_2 = M(2) + (E_1^2 - m_1^2)/M(2), \\ c_1 = 2M(2), \quad c_2 = [M(2)^2 + m_2^2 - m_1^2]/m_2.$$

Here we have introduced the energy E_1 of particle m_1 in the rest frame of m_2 [$p_1 = (E_1, \vec{p})$, $p_2 = (m_2, \vec{0})$, $M(2)^2 = (p_1 + p_2)^2$]:

$$E_1 = [M(2)^2 - (m_1^2 + m_2^2)]/2m_2. \quad (4.8)$$

The new equation satisfied by $\tilde{\phi}_2(0)$ is

$$(b_1 \Gamma_0 + b_2 \Gamma_3 - 2\alpha E_1) \tilde{\phi}_2(0) = 0,$$

or

$$\left[-2r_0 M(2) (r \nabla^2) - (E_1^2 - m_1^2) \frac{r}{2r_0 M(2)} - 2\alpha E_1 \right] \\ \times \tilde{\phi}_2(0) = 0. \quad (4.9)$$

Secondly we apply another scale transformation to get rid of the factors $2r_0 M(2)$ by putting

$$\tilde{\phi}_2(0) = e^{-i\delta M_3} \phi_{KG}(0), \quad (4.10)$$

with the tilting angle δ given by

$$\tanh\delta = [4M(2)^2 - 1]/[4M(2)^2 + 1].$$

The resultant equation,

$$\left[-r\gamma_0\nabla^2 - 2\alpha E_1 - \frac{\gamma}{r_0}(E_1^2 - m_1^2) \right] \phi_{\text{KG}}(0) = 0, \quad (4.11)$$

is a two-dimensional symmetric Klein-Gordon equation for particle m_1 moving in a $1/r$ potential field (after dividing with r , that is), in the approximation where the mass m_2 is much larger than m_1 . The spectrum of Eq. (4.11) is

$$E_1 = m_1 [1 + \alpha^2/(j + \frac{1}{2})^2]^{-1/2}, \quad (4.12)$$

where $(j + \frac{1}{2})$ is the spectrum of Γ_0 in the $\text{SO}(3,2)$ representation that we are using, with $j(j+1)$ being the eigenvalue of L^2 .

All we need now are a few identifications to pass to the generalized Majorana equation. Letting

$$r_0 = \frac{1}{2m_1} \left[1 + \frac{(j + \frac{1}{2})^2}{\alpha^2} \right]^{1/2} \quad (4.13)$$

and introducing M by the equation

$$\lambda M^2 + K = 2\alpha E_1 M r_0, \quad (4.14)$$

where K and λ are constants, we see from (4.12) that M also satisfies

$$(j + \frac{1}{2})M - \lambda M^2 - K = 0, \quad (4.15)$$

and (4.11) goes over into

$$\left(-r\nabla^2 - \frac{\lambda M^2 + K}{r_0 M} + \frac{\gamma}{4r_0^2} \right) \phi_{\text{KG}}(0) = 0, \quad (4.16)$$

or, with $\Gamma_0 = -r_0 r \nabla^2 + \gamma/4r_0$, into $[\phi_{\text{KG}}(0) \equiv \psi(0)]$

$$(\Gamma_0 M - \lambda M^2 - K)\psi(0) = 0. \quad (4.16')$$

This is the rest-frame generalized Majorana equation (2.6) and the mass spectrum of this equation is indeed given by (4.15). Transforming to a general frame we get Eq. (4.1). This completes the transformation of Eq. (4.1) into (4.2), or vice versa, the final form of which is thus, combining (4.3), (4.6), and (4.10),

$$\psi(p) = e^{i\vec{\xi} \cdot \vec{M}} e^{i(\delta + \eta)M_3} e^{-i\vec{\xi}_{(2)} \cdot \vec{P}_{(2)}} \cdot \vec{M}_{(2)} \phi_2(\vec{p}),$$

$$\vec{\xi} = \frac{\vec{p}}{|\vec{p}|} \sinh^{-1}(|\vec{p}|/M). \quad (4.17)$$

Note the distinction between the 2-vector $\vec{\xi}_{(2)}(\vec{P}_{(2)})$ and the 3-vector $\vec{\xi}(\vec{p})$.

We have made three separate transformations to exhibit various forms of the wave equation. The basic wave equation (4.4) in the rest frame can be solved directly by a single tilt operation

$$\left(M(2)^2 - \frac{[M(2)^2 + m_2^2 - m_1^2]^2}{4m_2^2} \right)^{1/2} \Gamma_0 |j\rangle$$

$$= \frac{\alpha}{2m_2} [M(2)^2 - m_1^2 - m_2^2] |j\rangle, \quad (4.18)$$

with

$$\vec{\phi}_2(j) \equiv |j\rangle \equiv e^{(-i\theta M_3)} \phi_2(0)$$

and the tilting angle

$$\tan\theta = -\frac{M(2)^2 + m_2^2 - m_1^2}{2m_2 M(2)}. \quad (4.19)$$

Because $\Gamma_0 |j\rangle = (j + \frac{1}{2}) |j\rangle$, the mass spectrum obtained from (4.18) is

$$M(2)^2 = m_1^2 + m_2^2 + 2m_1 m_2 [1 + \alpha^2/(j + \frac{1}{2})^2]^{-1/2}, \quad (4.20)$$

which is the same result as (4.8) and (4.12) combined. Equation (4.18) is equivalent to (4.11).

Thus starting from (4.1) we obtain (4.16') in the frame $P_\mu = (M, 0, 0, 0)$, and we obtain (4.2) in the frame $P_\mu = (M(2), 0, 0, P_3)$, with the identifications

$$\alpha_2 = -\lambda, \quad P^3 = \alpha_3 P_A P^A + \beta, \quad \gamma = -\lambda P_3 P^3 - K. \quad (4.21)$$

For the values of the coefficients given in (4.2) we get

$$P^3 = m_2 + m_1 (j + \frac{1}{2}) / [(j + \frac{1}{2})^2 + \alpha^2]^{1/2},$$

$$M = m_1 (j + \frac{1}{2}) / [(j + \frac{1}{2})^2 + \alpha^2]^{1/2}.$$

Hence for these values the parameter K is M -dependent but becomes a constant in the nonrelativistic limit¹³:

$$K = \frac{m_1 \alpha}{[1 + \alpha^2/(j + \frac{1}{2})^2]^{1/2}}$$

$$\times \left(1 - \frac{m_1}{2m_2} \frac{[\alpha/(j + \frac{1}{2})]^2}{[1 + \alpha^2/(j + \frac{1}{2})^2]^{1/2}} \right).$$

V. MINIMAL COUPLING OF THE COVARIANT WAVE EQUATION

In this section we shall couple the covariant wave equation (4.2) minimally to the electromagnetic field and show that the resulting matrix elements reduce to (3.11), i.e., those obtained from the two-dimensional dynamical picture of the system.

The interaction term from (4.2) is

$$-e \left(\Gamma_A - \frac{\alpha}{2m_2} \vec{P}_A + \frac{1}{2m_2} \Gamma_3 \vec{P}_A \right) A^A. \quad (5.1)$$

We take again a plane electromagnetic wave as in (3.4). Then the relativistic matrix elements between states $\phi_2(p)$ up to normalization constants are, using (4.18) and (4.3),

$$M_{\text{rel}} \propto \epsilon^A \int dr d\theta e^{i\vec{P}_{f(2)} \cdot \vec{r}_{(2)}} \bar{\phi}^*(j') e^{-i\theta(j') M_3} e^{-i\vec{\xi}_{f(2)} \cdot \vec{M}_{(2)}} \times \left[\Gamma_A - \frac{\alpha}{2m_2} (P_A^f + P_A^i) + \frac{1}{2m_2} \Gamma_3 (P_A^f + P_A^i) \right] e^{i\vec{\xi}_{i(2)} \cdot \vec{M}_{(2)}} e^{i\theta(j) M_3} \bar{\phi}(j) e^{-i\vec{P}_{i(2)} \cdot \vec{r}_{(2)}} e^{-i\vec{q}_{(2)} \cdot \vec{r}_{(2)}}, \quad (5.2)$$

where we have already integrated over external coordinates. (Compare similar situation in Ref. 7.)

We now evaluate the nonrelativistic limit of these matrix elements. The nonrelativistic limit of the boost parameter is

$$\vec{\xi}_{(2)}(\vec{P}_{(2)}) = \frac{\vec{P}_{(2)}}{|\vec{P}_{(2)}|} \sinh^{-1} \frac{|\vec{P}_{(2)}|}{M} \rightarrow \frac{\vec{P}_{(2)}}{M} \cong \frac{\vec{P}_{(2)}}{m_1 + m_2}. \quad (5.3)$$

Furthermore, from the commutation relations of M_i and Γ_i we have

$$e^{-i\theta M_3} \exp\left(-i \frac{\vec{P}_{(2)}}{m_1 + m_2} \cdot \vec{M}_{(2)}\right) e^{i\theta M_3} = \exp\left[-i \frac{\vec{P}_{(2)}}{m_1 + m_2} \cdot \cosh\theta(\vec{M}_{(2)} - C\vec{L}_{(2)} \tanh\theta)\right]. \quad (5.4)$$

Because in the nonrelativistic limit

$$\begin{aligned} \tanh\theta &\cong -1, \\ \cosh\theta &\cong \frac{(j + \frac{1}{2}) m_2 + m_1}{\alpha m_1}, \end{aligned}$$

the expression (5.4) becomes

$$\exp\left[-i \frac{(j + \frac{1}{2})}{\alpha m_1} \vec{P}_{(2)} \cdot \vec{r}_{(2)}\right]. \quad (5.4')$$

We also perform the transformation of the current occurring in (5.2). The second term drops out in the nonrelativistic limit and we get

$$\begin{aligned} e^{-i\theta(j') M_3} \left[\vec{\Gamma}_{(2)} + \frac{1}{2m_2} \Gamma_3 (\vec{P}_{f(2)} + \vec{P}_{i(2)}) \right] e^{-i\theta(j) M_3} \\ = \vec{\Gamma}_{(2)} + \frac{1}{2m_2} (\vec{P}_{f(2)} + \vec{P}_{i(2)}) \cosh\theta(j) \\ \times [\Gamma_3 + \Gamma_0 \tanh\theta(j)], \quad (5.5) \end{aligned}$$

which in the nonrelativistic limit is

$$\vec{\Gamma}_{(2)} + \frac{1}{2m_2} (\vec{P}_{f(2)} + \vec{P}_{i(2)}) \frac{(j + \frac{1}{2})}{\alpha} \frac{m_1 + m_2}{m_1 m_2} (\Gamma_3 - \Gamma_0), \quad (5.6)$$

and the matrix elements take the form

$$\begin{aligned} M_{\text{NR}} \propto \vec{\xi}_{(2)} \cdot \int dr d\theta e^{i(\vec{P}_{j(2)} - \vec{P}_{i(2)} - \vec{q}_{i(2)}) \cdot \vec{r}_{(2)}} \phi^*(j') \exp\left[-i \frac{j' + \frac{1}{2}}{\alpha m_1} (\vec{M}_{(2)} + C\vec{L}_{(2)}) \cdot \vec{P}_{f(2)}\right] \\ \times \left[\vec{\Gamma}_{(2)} + \frac{\vec{P}_{f(2)} + \vec{P}_{i(2)}}{2} \frac{m_1 + m_2}{m_1 m_2} \frac{(j' + \frac{1}{2})}{\alpha} (\Gamma_3 - \Gamma_0) \right] \\ \times \exp\left[i \frac{j' + \frac{1}{2}}{\alpha m_1} (\vec{M}_{(2)} + C\vec{L}_{(2)}) \cdot \vec{P}_{i(2)}\right] \exp\left[-i \ln\left(\frac{j + \frac{1}{2}}{j' + \frac{1}{2}}\right) M_3\right] \phi(j). \quad (5.7) \end{aligned}$$

This expression is indeed equivalent to (3.11), for using the relations

$$\psi = e^{i\theta M_3} \phi$$

for the states i and j , with

$$\theta = \ln\left(\frac{\alpha}{j + \frac{1}{2}} \frac{m_1 m_2}{m_1 + m_2}\right),$$

Eq. (3.11) can be rewritten as

$$\begin{aligned} M \propto \vec{\xi}_{(2)} \cdot \int dr d\theta \phi^*(j') e^{-i\theta_f M_3} \exp\left[i \vec{P}_{f(2)} \cdot \frac{m_2}{m_1 + m_2} 2r_0 (\vec{M}_{(2)} + C\vec{L}_{(2)})\right] \\ \times \left[\vec{\Gamma}_{(2)} + 2r_0 (\Gamma_0 - \Gamma_3) \frac{\vec{P}_{f(2)} + \vec{P}_{i(2)}}{2} \right] \exp\left[-i \vec{P}_{i(2)} \cdot \frac{m_2}{m_1 + m_2} 2r_0 (\vec{M}_{(2)} + C\vec{L}_{(2)})\right] e^{i\theta_i M_3} \phi(j). \end{aligned}$$

We transform the factors $e^{(-i\theta M_3)}$ to the right using (4.5) and obtain Eq. (5.7).

VI. CONCLUSIONS

The infinite-component Majorana equation in its rest frame describes a flat two-dimensional quantum system of two interacting particles in terms of some internal coordinates in an effective potential. We determine the covariant form (in two dimensions) of the motion of such a system as a whole, (Eq. 4.2), and show it to be equivalent to the (generalized) Majorana equation (4.1). The coupling of the covariant equation coincides with the coupling to the electromagnetic field of the flat two-body system. This has been shown in the Schrödinger (or Klein-Gordon) approximation, but the validity of this correspondence should be more general, as we know from our experience with the three-dimensional case. The coupling (5.1) can of course be reexpressed in terms of the equation

(4.1).

It should be remarked that there is no "derivation" of infinite-component wave equations from *relativistic* two-body dynamics. There are additional assumptions or postulates. This is true for the Bethe-Salpeter equation as well. One starts from a *nonrelativistic* internal dynamics and minimal coupling, then one puts the interaction into an algebraic form. Having recognized the dynamical group, one replaces the Galilean boosts by Lorentz boosts and arrives at the *postulate* of covariant infinite-component wave equation. The coefficients of the equations are determined by a correspondence argument. In this paper we have followed the reverse path and have investigated the internal motion corresponding to an *a priori* postulated infinite-component wave equation.

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