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Dynamical algebras of general Pöschl-Teller hierarchies

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Abstract. We investigate a class of operators connecting general Hamiltonians of the Pöschl-Teller type. The operators involved depend on three parameters and their explicit action on eigenfunctions is found. The whole set of intertwining operators close a $su(2,2) \approx so(4,2)$ Lie algebra. The space of eigenfunctions supports a differential-difference realization of an irreducible representation of the $su(2,2)$ algebra.

1. Introduction

The aim of this work is to find in a systematic way a general class of operators connecting Hamiltonians of the Pöschl-Teller class and to characterize their algebraic properties.

This is a problem that has been studied since a long time ago, and therefore one can find many partial results scattered in the literature. However, as we will see along the next sections, by approaching this problem in a quite general way we are able to find a complete dynamical algebra that was not considered up to now which closes a $su(2,2)$ Lie algebra structure.

Let us mention that the Hamiltonians we will deal with are related to the so called “shape-invariant potentials” [1], since some operators will change just the parameters in the same family of potentials. Sometimes the algebraic structure is called “potential algebra” [2], in order to be distinguished from the invariance algebras of a given Hamiltonian. Other type of operators are said to belong to “spectrum generating algebras” [3] since they may change also the energy eigenvalues of the Hamiltonians. A term wide enough to include any kind of intertwining operators in this work is “dynamical algebra” of the P-T hierarchy, since we are dealing with general operators connecting Hamiltonians in a given family. The operators conserving the value of the energy are termed “shift” operators (generating the potential algebra), while those changing the energy have a “ladder” flavour [4].

Here we will remind some references where one can find other approaches to this problem. The paper by Barut, Inomata and Wilson [5] worked the potential algebras of P-T Hamiltonians. More general dynamical algebras were considered in a series of papers by Quesne [6]. Other works concerned with spectrum generating operators of the P-T potentials are the book by Lange and Raab [7], and some others [8, 9]. In this paper we will follow closely the model of a previous work [10] devoted to the symmetric P-T potentials.

The contents of this work is the following. In section 2 we will summarize some results of standard factorizations [11] applied to the present case in order to get shift operators. Next,

in section 3 we will introduce a different kind of factorizations suitable to get ladder operators. Finally, in section 4 we will put together all the operators to define the dynamical algebra and we will end with some concluding remarks.

2. Standard shift-factorizations

We consider the following type of one-dimensional Pöschl-Teller Hamiltonians

$$H_{\alpha,\beta} = -\partial_\phi^2 + \frac{\alpha^2 - 1/4}{\cos^2 \phi} + \frac{\beta^2 - 1/4}{\sin^2 \phi}, \quad 0 < \phi < \pi/2 \quad (1)$$

where α, β are real constants.

2.1. A first set of “shift” operators, M^\pm

The Hamiltonian (1) is part of a “hierarchy of Hamiltonians” $\{H_{\alpha+n,\beta+n}\}$, $n \in \mathbb{Z}$, that has the following factorization property

$$H_{\alpha,\beta} = M_{\alpha,\beta}^+ M_{\alpha,\beta}^- + \mu_{\alpha,\beta} = M_{\alpha-1,\beta-1}^- M_{\alpha-1,\beta-1}^+ + \mu_{\alpha-1,\beta-1} \quad (2)$$

where

$$M_{\alpha,\beta}^\pm = \pm \partial_\phi - (\alpha + 1/2) \tan \phi + (\beta + 1/2) \cot \phi \quad (3)$$

and

$$\mu_{\alpha,\beta} = (\alpha + \beta + 1)^2 \quad (4)$$

The discrete eigenvalues of each Hamiltonian $H_{\alpha,\beta}$ are related with the ground energy of the hierarchy $\{H_{\alpha+n,\beta+n}\}$ as follows. We will consider the ground states annihilated by the operators $M_{\alpha,\beta}^-$,

$$M_{\alpha,\beta}^- \psi_{(\alpha,\beta)}^0 = 0 \Rightarrow \psi_{(\alpha,\beta)}^0 = K_{(\alpha,\beta)}^0 \cos \phi^{\alpha+1/2} \sin \phi^{\beta+1/2} \quad (5)$$

where $K_{(\alpha,\beta)}^0$ is a normalizing constant. The corresponding energy eigenvalue is given, according to (2), by

$$E_{(\alpha,\beta)}^0 = \mu_{\alpha,\beta} = (\alpha + \beta + 1)^2 \quad (6)$$

The action of these factor operators on eigenfunctions of the Hamiltonian hierarchy derived from the intertwining relation (2), up to normalization coefficients, is

$$\begin{aligned} M_{\alpha,\beta}^- : \psi_{(\alpha,\beta)}^n &\rightarrow \psi_{(\alpha+1,\beta+1)}^{n-1} \\ M_{\alpha,\beta}^+ : \psi_{(\alpha+1,\beta+1)}^{n-1} &\rightarrow \psi_{(\alpha,\beta)}^n \end{aligned} \quad (7)$$

where $\psi_{(\alpha,\beta)}^n$ denotes an eigenfunction of $H_{\alpha,\beta}$ corresponding to the eigenvalue $E_{(\alpha,\beta)}^n$ and n is for the energy level of the corresponding Hamiltonian. We say that they are “shift operators” that change the potential parameters but keep the value of energy since

$$E_{(\alpha,\beta)}^n = E_{(\alpha+n,\beta+n)}^0 = (\alpha + \beta + 2n + 1)^2, \quad n = 0, 1, 2, \dots \quad (8)$$

As the differential operators $M_{\alpha,\beta}^\pm$ are Hermitian conjugated, we can easily obtain the coefficients of their action

$$M_{\alpha-1,\beta-1}^+ \psi_{(\alpha,\beta)}^n = 2\sqrt{n(\alpha + \beta + n - 1)} \psi_{(\alpha-1,\beta-1)}^{n+1} \quad (9)$$

and

$$M_{\alpha-1,\beta-1}^- \psi_{(\alpha-1,\beta-1)}^{n+1} = 2\sqrt{n(\alpha + \beta + n - 1)} \psi_{(\alpha,\beta)}^n \quad (10)$$

We can define “free index operators” $\tilde{M}^\pm = \frac{1}{2}M^\pm$ connecting eigenfunctions of consecutive Hamiltonians in this hierarchy according to (7), so that the relevant commutators have the form

$$[\tilde{M}^-, \tilde{M}^+] = -2\tilde{M}, \quad [\tilde{M}, \tilde{M}^\pm] = \pm\tilde{M}^\pm \quad (11)$$

and where the diagonal operator \tilde{M} is defined by

$$\tilde{M}\psi_{(\alpha,\beta)}^n = -\frac{1}{2}(\alpha + \beta + 2n)\psi_{(\alpha,\beta)}^n \quad (12)$$

Therefore these “tilde operators” span a $su(2)$ Lie algebra.

Henceforth it will be convenient to introduce also a “three-subindex” notation for eigenfunctions,

$$\begin{aligned} H_{\alpha,\beta}\psi_{(\alpha,\beta,\epsilon)} &= E_\epsilon\psi_{(\alpha,\beta,\epsilon)} \\ E_\epsilon &= \epsilon^2, \quad \psi_{(\alpha,\beta,\epsilon)} \equiv \psi_{(\alpha,\beta)}^n, \quad \epsilon = \alpha + \beta + 2n + 1 \end{aligned} \quad (13)$$

From the factorization properties (2) and making use of the new notation, the action of the \tilde{M}^\pm operators can be expressed in the form

$$\begin{cases} \tilde{M}^-\psi_{(\alpha,\beta,\epsilon)} = \frac{1}{2}\sqrt{(\epsilon + \alpha + \beta + 1)(\epsilon - \alpha - \beta - 1)}\psi_{(\alpha+1,\beta+1,\epsilon)} \\ \tilde{M}^+\psi_{(\alpha+1,\beta+1,\epsilon)} = \frac{1}{2}\sqrt{(\epsilon + \alpha + \beta + 1)(\epsilon - \alpha - \beta - 1)}\psi_{(\alpha,\beta,\epsilon)} \\ \tilde{M}\psi_{(\alpha,\beta,\epsilon)} = -\frac{1}{2}(\alpha + \beta)\psi_{(\alpha,\beta,\epsilon)} \end{cases} \quad (14)$$

In this notation it is evident that these operators change the potential parameters α, β but leave the energy parameter ϵ fixed.

2.2. Second set of shift operators: N^\pm

In a similar way we find another set of factorization operators,

$$H_{\alpha,\beta} = N_{\alpha,\beta}^+ N_{\alpha,\beta}^- + \nu_{\alpha,\beta} = N_{\alpha+1,\beta-1}^- N_{\alpha+1,\beta-1}^+ + \nu_{\alpha+1,\beta-1} \quad (15)$$

where

$$N_{\alpha,\beta}^\pm = \pm\partial_\phi + (\alpha - 1/2)\tan\phi + (\beta + 1/2)\cot\phi \quad (16)$$

and

$$\nu_{\alpha,\beta} = (-\alpha + \beta + 1)^2 \quad (17)$$

This new set of operators leads to the hierarchy $\{H_{\alpha-m,\beta+m}\}$ with eigenvalues

$$E_{\alpha,\beta}^m = E_{\alpha-m,\beta+m}^0 = (-\alpha + \beta + 2m + 1)^2 \quad (18)$$

The action of the new factor operators on the eigenfunctions of this hierarchy is as follows

$$\begin{aligned} N_{\alpha,\beta}^- : \tilde{\psi}_{(\alpha,\beta)}^m &\rightarrow \tilde{\psi}_{(\alpha-1,\beta+1)}^{m-1} \\ N_{\alpha,\beta}^+ : \tilde{\psi}_{(\alpha-1,\beta+1)}^{m-1} &\rightarrow \tilde{\psi}_{(\alpha,\beta)}^m \end{aligned} \quad (19)$$

They are also shift operators that change the potential parameters but keep the value of energy:

$$E_{(\alpha,\beta)}^m = E_{(\alpha-1,\beta+1)}^{m-1}.$$

Free index operators can be introduced together with rescaling factors:

$$\tilde{N}^\pm = \frac{1}{2}N^\pm, \quad \tilde{N}\tilde{\psi}_{(\alpha,\beta)}^n = -\frac{1}{2}(-\alpha + \beta + 2n)\tilde{\psi}_{(\alpha,\beta)}^n \quad (20)$$

so that we get another $su(2)$ Lie algebra,

$$[\tilde{N}, \tilde{N}^\pm] = \pm \tilde{N}^\pm, \quad [\tilde{N}^-, \tilde{N}^+] = -2\tilde{N} \quad (21)$$

The explicit action of these operators on the eigenfunctions $\psi_{(\alpha,\beta,\epsilon)}$ is

$$\begin{cases} \tilde{N}^- \tilde{\psi}_{(\alpha,\beta,\epsilon)} = \frac{1}{2} \sqrt{(\epsilon - \alpha + \beta + 1)(\epsilon + \alpha - \beta - 1)} \tilde{\psi}_{(\alpha-1,\beta+1,\epsilon)} \\ \tilde{N}^+ \tilde{\psi}_{(\alpha-1,\beta+1,\epsilon)} = \frac{1}{2} \sqrt{(\epsilon - \alpha + \beta + 1)(\epsilon + \alpha - \beta - 1)} \tilde{\psi}_{(\alpha,\beta,\epsilon)} \\ \tilde{N} \tilde{\psi}_{(\alpha,\beta,\epsilon)} = -\frac{1}{2}(-\alpha + \beta) \tilde{\psi}_{(\alpha,\beta,\epsilon)} \end{cases} \quad (22)$$

We have checked that the two $su(2)$ Lie algebras introduced up to now spanned by the generators $\{\tilde{M}^\pm, \tilde{M}\}$ and $\{\tilde{N}^\pm, \tilde{N}\}$ commute, leading to a direct sum $su(2) \oplus su(2)$. The unitary representations compatible with the conditions on the parameters α, β are just “square representations”, $j \otimes j$, with $2j = n \in \mathbb{Z}^+$, and dimension n^2 .

3. Ladder factorizations

3.1. Ladder X^\pm and W^\pm operators

In order to get ladder operators that change energy eigenvalues, we will start from the initial Hamiltonian with parameters (α, β) applied to an eigenfunction $\psi_{(\alpha,\beta)}^n$ with eigenvalue given in (8)

$$H_{\alpha,\beta} \psi_{(\alpha,\beta)}^n = E_{(\alpha,\beta)}^n \psi_{(\alpha,\beta)}^n = (\alpha + \beta + 2n + 1)^2 \psi_{(\alpha,\beta)}^n \quad (23)$$

Here, it will be helpful to use the “three subindex notation” valid for general eigenvalues:

$$H_{\alpha,\beta} \psi_{(\alpha,\beta,\epsilon)} = E_\epsilon \psi_{(\alpha,\beta,\epsilon)} = \epsilon^2 \psi_{(\alpha,\beta,\epsilon)} \quad (24)$$

Now, we want to change the value of β and conserve that of α , so we will multiply the equation $H_{\alpha,\beta} \psi_{(\alpha,\beta,\epsilon)} = E_\epsilon \psi_{(\alpha,\beta,\epsilon)}$ by $\cos^2 \phi$, to obtain

$$(-\cos^2 \phi \partial_\phi^2 + (\alpha^2 - 1/4) + (\beta^2 - 1/4) \cot^2 \phi) \psi_{(\alpha,\beta)}^n = \epsilon^2 \cos^2 \phi \psi_{(\alpha,\beta)}^n \quad (25)$$

and rewrite it in the form

$$(-\cos^2 \phi \partial_\phi^2 + (\beta^2 - 1/4) \cot^2 \phi - \epsilon^2 \cos^2 \phi) \psi_{(\alpha,\beta)}^n = -(\alpha^2 - 1/4) \psi_{(\alpha,\beta)}^n \quad (26)$$

For simplicity we introduce the notation $h_{\beta,\epsilon}$ for the relevant differential operator of the previous equation

$$h_{\beta,\epsilon} = -\cos^2 \phi \partial_\phi^2 + (\beta^2 - 1/4) \cot^2 \phi - \epsilon^2 \cos^2 \phi \quad (27)$$

thus, (26) becomes

$$h_{\beta,\epsilon} \psi_{(\alpha,\beta,\epsilon)} = -(\alpha^2 - 1/4) \psi_{(\alpha,\beta,\epsilon)} \quad (28)$$

Then, we are able to factorize the differential operator (27) in the usual way

$$h_{\beta,\epsilon} = X_{\beta,\epsilon}^+ X_{\beta,\epsilon}^- + \xi_{\beta,\epsilon} = X_{\beta-1,\epsilon-1}^- X_{\beta-1,\epsilon-1}^+ + \xi_{\beta-1,\epsilon-1} \quad (29)$$

where

$$X_{\beta,\epsilon}^\pm = \pm \cos \phi \partial_\phi + \frac{\beta + 1/2}{\sin \phi} + (\epsilon + 1/2 \pm 1/2) \sin \phi \quad (30)$$

and

$$\xi_{\beta,\epsilon} = -(\beta + \epsilon + 3/2)(\beta + \epsilon + 1/2) \quad (31)$$

Therefore, the action of these ladder operators on eigenfunctions is as follows

$$\begin{aligned} X_{\beta,\epsilon}^- &: \psi_{(\alpha,\beta,\epsilon)} \rightarrow \psi_{(\alpha,\beta+1,\epsilon+1)} \\ X_{\beta,\epsilon}^+ &: \psi_{(\alpha,\beta+1,\epsilon+1)} \rightarrow \psi_{(\alpha,\beta,\epsilon)} \end{aligned} \quad (32)$$

Since in this relation $E_\epsilon \neq E_{\epsilon\pm 1}$ (or $E_{(\alpha,\beta)}^n \neq E_{(\alpha,\beta\pm 1)}^n$), we are dealing with ladder operators in the sense that they change the energy eigenvalue (as well as other parameters of the potential) of the eigenfunctions of the associated Hamiltonian hierarchy.

Starting from \tilde{X}^\pm, \tilde{X} , we can find Hermitian conjugated differential operators by means of suitable coefficients [12] \tilde{X}^\pm, \tilde{X} given by

$$\tilde{X}_{\alpha,\beta,\epsilon}^- = \frac{1}{2} \sqrt{\frac{\epsilon+1}{\epsilon}} X_{\alpha,\beta,\epsilon}^-, \quad \tilde{X}_{\alpha,\beta,\epsilon}^+ = \frac{1}{2} \sqrt{\frac{\epsilon}{\epsilon+1}} X_{\alpha,\beta,\epsilon}^+ \quad (33)$$

and their action on normalized eigenfunctions is

$$\begin{cases} \tilde{X}^- \psi_{(\alpha,\beta,\epsilon)} = \frac{1}{2} \sqrt{(\beta+\epsilon+\alpha+1)(\beta+\epsilon-\alpha+1)} \psi_{(\alpha,\beta+1,\epsilon+1)} \\ \tilde{X}^+ \psi_{(\alpha,\beta+1,\epsilon+1)} = \frac{1}{2} \sqrt{(\beta+\epsilon+\alpha+1)(\beta+\epsilon-\alpha+1)} \psi_{(\alpha,\beta,\epsilon)} \\ \tilde{X} \psi_{(\alpha,\beta,\epsilon)} = -\frac{1}{2}(\beta+\epsilon) \psi_{(\alpha,\beta,\epsilon)} \end{cases} \quad (34)$$

There is a second factorization of the differential operator (27), that we write as follows [12]

$$W_{\beta,\epsilon}^\pm = \pm \cos \phi \partial_\phi - \frac{\beta-1/2}{\sin \phi} + (\epsilon+1/2 \pm 1/2) \sin \phi \quad (35)$$

so that

$$h_{\beta,\epsilon} = W_{\beta,\epsilon}^+ W_{\beta,\epsilon}^- + \omega_{\beta,\epsilon} = W_{\beta+1,\epsilon-1}^- W_{\beta+1,\epsilon-1}^+ + \omega_{\beta+1,\epsilon-1} \quad (36)$$

and

$$\omega_{\beta,\epsilon} = -(-\beta+\epsilon+3/2)(-\beta+\epsilon+1/2) \quad (37)$$

As before, the action of the Hermitian conjugated operators \tilde{W}^\pm, \tilde{W} has the following form

$$\begin{cases} \tilde{W}^- \psi_{(\alpha,\beta,\epsilon)} = \frac{1}{2} \sqrt{(-\beta+\epsilon+\alpha+1)(-\beta+\epsilon-\alpha+1)} \psi_{(\alpha,\beta-1,\epsilon+1)} \\ \tilde{W}^+ \psi_{(\alpha,\beta-1,\epsilon+1)} = \frac{1}{2} \sqrt{(-\beta+\epsilon+\alpha+1)(-\beta+\epsilon-\alpha+1)} \psi_{(\alpha,\beta,\epsilon)} \\ \tilde{W} \psi_{(\alpha,\beta,\epsilon)} = -\frac{1}{2}(-\beta+\epsilon) \psi_{(\alpha,\beta,\epsilon)} \end{cases} \quad (38)$$

3.2. Ladder Y^\pm and Z^\pm operators

Next, we will obtain ladder operators changing the value of α and maintaining the value of β . To this end, we will multiply the eigenvalue equation (23) by $\sin^2 \phi$,

$$(-\sin^2 \phi \partial_\phi^2 + (\alpha^2 - 1/4) \tan^2 \phi + (\beta^2 - 1/4)) \psi_{(\alpha,\beta)}^n = E_{(\alpha,\beta)}^n \sin^2 \phi \psi_{(\alpha,\beta)}^n \quad (39)$$

We can make use of the “three subindex notation”, with the third subindex ϵ , by starting with the redefinition of the differential operator in (39) as

$$g_{\alpha,\epsilon} = -\sin^2 \phi \partial_\phi^2 + (\alpha^2 - 1/4) \tan^2 \phi - \epsilon^2 \sin^2 \phi \quad (40)$$

Then, we can find the factorization

$$g_{\alpha,\epsilon}^n = Y_{\alpha,\epsilon}^+ Y_{\alpha,\epsilon}^- + \eta_{\alpha,\epsilon} = Y_{\alpha-1,\epsilon-1}^- Y_{\alpha-1,\epsilon-1}^+ + \eta_{\alpha-1,\epsilon-1} \quad (41)$$

with

$$Y_{\alpha,\epsilon}^\pm = \mp \sin \phi \partial_\phi + \frac{\alpha + 1/2}{\cos \phi} + (\epsilon + 1/2 \pm 1/2) \cos \phi \quad (42)$$

and

$$\eta_{\alpha,\epsilon} = -(\alpha + \epsilon + 3/2)(\alpha + \epsilon + 1/2). \quad (43)$$

The action of the corresponding Hermitian conjugated operators [12] on normalized $\psi_{(\alpha,\beta,\epsilon)}$ eigenfunctions is

$$\begin{cases} \tilde{Y}^- \psi_{(\alpha,\beta,\epsilon)} = \frac{1}{2} \sqrt{(\alpha + \epsilon + \beta + 1)(\alpha + \epsilon - \beta + 1)} \psi_{(\alpha+1,\beta,\epsilon+1)} \\ \tilde{Y}^+ \psi_{(\alpha+1,\beta,\epsilon+1)} = \frac{1}{2} \sqrt{(\alpha + \epsilon + \beta + 1)(\alpha + \epsilon - \beta + 1)} \psi_{(\alpha,\beta,\epsilon)} \\ \tilde{Y} \psi_{(\alpha,\beta,\epsilon)} = -\frac{1}{2}(\alpha + \epsilon) \psi_{(\alpha,\beta,\epsilon)} \end{cases} \quad (44)$$

Finally, we will define a last set of ladder operators by factorizing (40) as

$$g_{\alpha,\epsilon} = Z_{\alpha,\epsilon}^+ Z_{\alpha,\epsilon}^- + \zeta_{\alpha,\epsilon}^n = Z_{\alpha+1,\epsilon-1}^- Z_{\alpha+1,\epsilon-1}^+ + \zeta_{\alpha+1,\epsilon-1} \quad (45)$$

where

$$\zeta_{\alpha,\epsilon} = -(-\alpha + \epsilon + 3/2)(-\alpha + \epsilon + 1/2) \quad (46)$$

The Hermitian conjugated operators \tilde{Z}^\pm can be defined in same way as the \tilde{Y}^\pm operators. Their action is

$$\begin{cases} \tilde{Z}^- \psi_{(\alpha,\beta,\epsilon)} = \frac{1}{2} \sqrt{(-\alpha + \epsilon + \beta + 1)(-\alpha + \epsilon - \beta + 1)} \psi_{(\alpha-1,\beta,\epsilon+1)} \\ \tilde{Z}^+ \psi_{(\alpha-1,\beta,\epsilon+1)} = \frac{1}{2} \sqrt{(-\alpha + \epsilon + \beta + 1)(-\alpha + \epsilon - \beta + 1)} \psi_{(\alpha,\beta,\epsilon)} \\ \tilde{Z} \psi_{(\alpha,\beta,\epsilon)} = -\frac{1}{2}(-\alpha + \epsilon) \psi_{(\alpha,\beta,\epsilon)} \end{cases} \quad (47)$$

4. Complete dynamical algebra

Now, we will consider the whole set of lowering and raising operators of the previous sections together: $\{\tilde{M}^\pm, \tilde{N}^\pm, \tilde{X}^\pm, \tilde{W}^\pm, \tilde{Y}^\pm, \tilde{Z}^\pm\}$, including those having a “shift” or “ladder” character. The diagonal operators $\{\tilde{M}, \tilde{N}, \tilde{X}, \tilde{W}, \tilde{Y}, \tilde{Z}\}$ can be expressed as linear combinations of three independent diagonal operators $\{D_\alpha, D_\beta, D_\epsilon\}$ defined by

$$D_\alpha \psi_{(\alpha,\beta,\epsilon)} = \alpha \psi_{(\alpha,\beta,\epsilon)}, \quad \text{etc.} \quad (48)$$

Therefore, in all we have 15 independent operators closing the following commutations relations:

$$\begin{aligned} [\tilde{X}^-, \tilde{M}^+] &= -\tilde{Z}^-, & [\tilde{X}^-, \tilde{N}^+] &= -\tilde{Y}^-, & [\tilde{W}^+, \tilde{M}^+] &= -\tilde{Y}^+, \\ [\tilde{W}^+, \tilde{N}^+] &= -\tilde{Z}^+, & [\tilde{Y}^+, \tilde{N}^+] &= \tilde{X}^+, & [\tilde{Y}^-, \tilde{M}^+] &= \tilde{W}^-, \\ [\tilde{Y}^-, \tilde{X}^+] &= -\tilde{N}^+, & [\tilde{Y}^-, \tilde{W}^+] &= \tilde{M}^-, & [\tilde{Z}^+, \tilde{M}^+] &= \tilde{X}^+, \\ [\tilde{Z}^-, \tilde{N}^+] &= \tilde{W}^-, & [\tilde{Z}^-, \tilde{X}^+] &= -\tilde{M}^+, & [\tilde{Z}^-, \tilde{W}^+] &= \tilde{N}^-, \end{aligned} \quad (49)$$

where we have included only some non-vanishing commutators of different sets. These relations correspond to the $so(4, 2) \approx su(2, 2)$ Lie algebra, including the subalgebra $su(2) \oplus su(2)$ generated by shift operators.

We will end this work by remarking some features of our dynamical algebra.

- We have proposed a general factorization scheme in the line of [13] where there are three parameters: α and β for potentials and ϵ for energies (for more details see [12]).
- We have obtained a general solution for the intertwining operators under these general conditions.
- Different operators are related by means of reflections as it has been shown in [12].
- The whole set of intertwining operators close a Lie algebra $su(2, 2)$. Other approaches leading to intertwining operators can be seen in [14, 15, 16].
- From this intertwining operators we can get ‘pure’ ladder operators leading to an spectrum generating algebra [12]. These operators have a classical analogue closing a classical spectrum generating algebra [17], which is relevant in studying the properties of coherent states [18, 19].

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References

- [1] Gendenshtein L 1983 *JETP Lett.* **38** 356
- [2] Alhassid Y, Gürsey F and Iachello F 1983 *Ann. Phys.* **148** 346
- [3] Gadella M, Negro J, Nieto L M, Pronko G P and Santander M 2011 *J. Math. Phys.* **52** 063509
- [4] Cooper I L 1983 *J. Phys. A: Math. Gen.* **26** 1601
- [5] Barut A O, Inomata A and Wilson R 1987 *J. Phys. A: Math. Gen.* **20** 4075; 1987 *J. Phys. A: Math. Gen.* **20** 4083
- [6] Quesne C 1988 *J. Phys. A: Math. Gen.* **21** 4487; 1988 *J. Phys. A: Math. Gen.* **21** 4501; 1989 *J. Phys. A: Math. Gen.* **22** 3723
- [7] De Lange O L and Raab R E 1991 *Operator Methods in Quantum Mechanics* (Oxford: Oxford University Press)
- [8] Dong S-H and Lemus R 2001 *Int. J. Quantum Chem.* **86** 265
- [9] Dong S-H, Sun G-H and Tang Y 2003 *Int. J. Mod. Phys. E* **12** 809
- [10] Kuru Ş, Negro J 2009 *Ann. Phys.* **324** 2548
- [11] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
- [12] Kuru Ş, Negro J 2011 *Dynamical algebras of general two-parametric Pöschl-Teller Hamiltonians* submitted for publication elsewhere
- [13] Negro J, Nieto L M and Rosas-Ortiz O 2000 *J. Phys. A: Math. Gen.* **33** 7207
- [14] Liu Y F, Lei Y A and Zeng J Y 1997 *Phys. Lett. A* **231** 9
- [15] Xu B-W and Kong F E 1999 *Phys. Lett. A* **259** 212
- [16] Cotfas N 2002 *J. Phys. A: Math. Gen.* **35** 9355
- [17] Kuru Ş, Negro J 2008 *Ann. Phys.* **323** 413
- [18] Nieto M M and Simmons L M 1979 *Phys. Rev. D* **20** 1332
- [19] Cruz y Cruz S, Kuru Ş and Negro J 2008 *Phys. Lett. A* **372** 1391