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Integrals of motion and trajectories for the Perlick system I: an algebraic approach

O Ragnisco¹, S Kuru², J Negro³

¹ Istituto Nazionale di Fisica Nucleare sezione di Roma Tre, Via Vasca Navale 84, I-00146 Roma, Italy

² Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey

³ Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47011 Valladolid, Spain

E-mail: ragnisco@fis.uniroma3.it, kuru@science.ankara.edu.tr, jnegro@fta.uva.es

Abstract. In this paper we tersely recall the main algebraic and geometric properties of the maximally superintegrable system known as “Perlick System Tipe I”, considering all possible values of the relevant parameters. We will follow a classical variant of the so called factorization method, emphasizing the role played the Poisson Algebra of the constants of motion in shedding light on the geometric features of the trajectories.

1. Introduction

In his famous paper dating back to 1992 ([1]), V. Perlick identified two multi-parametric families of Lagrangean systems on curved manifolds complying with the requirements of Bertrand’s Theorem ([2]), namely (i) the existence of stable circular orbits, and (ii) the fact that all bounded trajectories are closed.

The two families are deformation either of the Kepler-Coulomb system (Family I) or of the isotropic harmonic oscillator (Family II), the corresponding classical Bertrand systems being recovered in the Euclidean limit. Moreover both families can be extended to an arbitrary number of dimensions by (hyper) spherical symmetry, and are amenable to ‘an exact’ quantum-mechanical treatment.

It turns out that Perlick Systems are the most general example of (hyper-)spherically symmetric “Maximally Superintegrable” hamiltonian systems ([3]). Indeed, in the case of N degrees of freedom, they are equipped with the maximum number $(2N - 1)$ of independent integrals of motion. Such extra integrals are of course related to *additional* dynamical symmetries. They are associated to the *Laplace-Runge-Lenz* vector in the case of Family (I) and to the *Demkov-Fradkin* tensor [4],[5], [6] in the case of Family (II). Moreover, maximal superintegrability entails a further crucial property, namely that *the trajectories can be determined without resorting to analytic calculations, thanks to the underlying symmetry algebra*

Here we restrict our analysis to Family (I). Most of the results will be presented in a sketchy way; for further details we refer the interested reader to the letter [8].



2. Factorization Method in a Classical Context

2.1. The Perlick Hamiltonian: radial and angular parts

We recall Perlick I Hamiltonian in spherical coordinates (r, θ, φ) :

$$\tilde{H}^{\pm} = \beta^2(1 + K r^2) \frac{p_r^2}{2} + \frac{\mathbf{L}^2}{2r^2} + G \pm \frac{1}{r} \sqrt{1 + K r^2}, \quad (2.1)$$

where \mathbf{L} is the angular momentum and \mathbf{L}^2 can be considered as an angular Hamiltonian $H_{\theta\varphi}$ in the variables (θ, φ) , having the form

$$\mathbf{L}^2 = H_{\theta\varphi} = p_{\theta}^2 + \frac{p_{\varphi}^2}{\sin^2 \theta} \quad (2.2)$$

being K, G real constants and $\beta = m/n$ a rational number.

We notice that p_{φ}^2 can be seen as a free Hamiltonian H_{φ} in the variables (φ, p_{φ}) , defined on the unit circle. The radial variable has different ranges depending on the sign of K . Indeed, If $K \geq 0$, then $0 < r < \infty$, while if $K < 0$, r must be restricted to the finite interval $0 < r < 1/\sqrt{-K}$. The Perlick choice was \tilde{H}^{-} , to include bounded motion, but in the case $K < 0$ we need both Hamiltonians \tilde{H}^{\pm} to have a well defined Hamiltonian on the full three dimensional sphere. The associated metric reads:

$$ds^2 = \frac{dr^2}{\beta^2(1 + K r^2)} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.3)$$

The “radial” hamiltonian \tilde{H}_r^{\pm} is simply obtained by setting in 2.1 $G = 0$, and letting $\mathbf{L}^2 \rightarrow \ell^2$:

$$\tilde{H}_r^{\pm} = \beta^2(1 + K r^2) \frac{p_r^2}{2} + \frac{\ell^2}{2r^2} \pm \frac{1}{r} \sqrt{1 + K r^2}. \quad (2.4)$$

Since $\{p_{\varphi}, \tilde{H}^{\pm}\} = 0$, p_{φ} is another constant of motion: $p_{\varphi} = \ell_z = \text{const.}$ Then from the angular Hamiltonian $H_{\theta\varphi}$ we get the effective polar Hamiltonian H_{θ} depending on the single variable θ :

$$H_{\theta} = p_{\theta}^2 + \frac{\ell_z^2}{\sin^2 \theta}.$$

Notice that H_{θ} is singular at the angles $\theta = 0$ and $\theta = \pi$, so the trajectories lie between these two values (i.e. the North and South poles cannot be reached unless $\ell_z = 0$). Once fixed the values of ℓ, ℓ_z , the turning points for θ are given by the solutions of $\ell^2 = \ell_z^2 / \sin^2 \theta$.

2.2. New Coordinates

To describe both signs of K in a unified way we introduce κ -dependent trigonometric functions:

$$C_{\kappa}(u) \equiv \begin{cases} \cos \sqrt{\kappa} u & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh \sqrt{-\kappa} u & \kappa < 0 \end{cases}, \quad S_{\kappa}(u) \equiv \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} u & \kappa > 0 \\ u & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} u & \kappa < 0 \end{cases}.$$

The κ -tangent is defined by

$$T_{\kappa}(u) \equiv \frac{S_{\kappa}(u)}{C_{\kappa}(u)}.$$

Useful relations among these κ -functions are:

$$C_{\kappa}^2(u) + \kappa S_{\kappa}^2(u) = 1, \quad C'_{\kappa}(u) = -\kappa S_{\kappa}(u), \quad S'_{\kappa}(u) = C_{\kappa}(u).$$

Let us define:

$$K = -\kappa, \quad r = S_{\kappa}(\xi), \quad p_r = \frac{p_{\xi}}{C_{\kappa}(\xi)},$$

where the range of ξ is as follows:

- for $\kappa \leq 0$, $0 < \xi < \infty$;
- for $\kappa > 0$, we have two natural choices: $0 < \xi < \pi/(2\sqrt{\kappa})$ or $\pi/(2\sqrt{\kappa}) < \xi < \pi/\sqrt{\kappa}$.

The variables (ξ, θ, φ) are angular coordinates of the points of a (pseudo) sphere in \mathbb{R}^4 .

- when $\kappa < 0$ they parametrize the points of one sheet of a three dimensional (3D) hyperboloid;
- when $\kappa = 0$ the coordinates (ξ, θ, φ) represent the spherical coordinates of the points of \mathbb{R}^3 ;
- finally, for $\kappa > 0$ these variables with $0 < \xi < \pi/(2\sqrt{\kappa})$ parametrize the points of one half (the north hemisphere) of a deformed 3D sphere, and the values $\pi/(2\sqrt{\kappa}) < \xi < \pi/\sqrt{\kappa}$ give the points of the south hemisphere.

In the Perlick's original case, $\kappa < 0$, this choice of coordinates is not important since the north and south hemispheres of a hyperboloid are not connected.

We end this subsection by making a few remarks about terminology.

In the sequel, we will speak about *shift* and *ladder* functions, with some abuse of language. It is a terminology coming from the quantum context that we import by analogy to the classical setting. A good description can be found for instance in [9].

In a not too rigorous way, in the language of Supersymmetric Quantum Mechanics, *shift* operators are those associated with a factorization of a second order differential operator (typically the Schroedinger operator) such that, by interchanging the factors, we pass from a given potential to a novel one having the same functional form (*shape*) but a different (i.e. *shifted*) coupling constant. In principle the process can be indefinitely iterated, yielding a sequence of *almost isospectral* operators (at each stage we “miss” the ground state). Conversely, *ladder* operators (as the usual “raising” and “lowering” operators) do not change the original Schroedinger operator, but act on each eigenstate transforming it in a new one associated to a higher (or lower) eigenvalue.

3. Constants of Motion

In this model, we already have three independent constants of motion: $H, H_{\theta\varphi}, H_{\varphi}$ with constant values given by E, ℓ^2, ℓ_z^2 . Our aim is to find additional constants of motion by means of the factorization method in order to show the maximal superintegrability of the system. We recall that the application of the factorisation method to the classical realm was first proposed in [12] and then developed in [13], [14], [15], [16], [17], [18]. A first pair of constants X^{\pm} will be obtained from the effective Hamiltonians H_{ξ} and H_{θ} , and a second pair Y^{\pm} from the next two effective Hamiltonians, H_{θ} and H_{φ} .

- The constants of motion X^{\pm}

The first set of constants X^{\pm} are constructed in terms of the shift functions of H_{ξ} and the ladder functions of H_{θ} which are obtained as follows.

- *Shift functions of H_{ξ}*

The Hamiltonian H_{ξ} can be factorized as

$$H_{\xi} = B^{+}B^{-} + \lambda_{\xi},$$

where

$$B^{\pm} = \frac{1}{\sqrt{2}} \left(\mp i \beta p_{\xi} + \frac{\ell}{T_{\kappa}(\xi)} - \frac{1}{\ell} \right), \quad \lambda_{\xi} = -\frac{1}{2} \left(\frac{1}{\ell^2} - \kappa \ell^2 \right).$$

The shift functions B^{\pm} are complex conjugate of each other and they satisfy the following PBs together with the Hamiltonian H_{ξ}

$$\{B^{-}, B^{+}\} = i \beta \frac{\ell}{S_{\kappa}^2(\xi)}, \quad \{H_{\xi}, B^{\pm}\} = \pm i \beta \frac{\ell}{S_{\kappa}^2(\xi)} B^{\pm}.$$

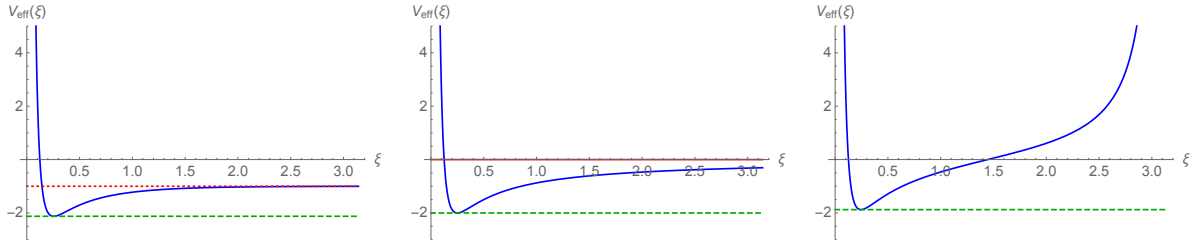


Figure 1. Plot of the effective potential $V_{\text{eff}}(\xi)$ for $\kappa = -1$ (left), $\kappa = 0$ (center) and $\kappa = 1$ (right) together with the limiting values of the energy for $\ell = 0.5$. The range of κ for $\kappa \leq 0$ is $0 < \xi < \infty$ and for $\kappa > 0$ it is $0 < \xi < \pi/\sqrt{\kappa}$.

The second PB implies that

$$\{H, B^{\pm}\} = \pm i \beta \frac{\sqrt{H_{\theta\varphi}}}{S_{\kappa}^2(\xi)} B^{\pm}.$$

From the above factorization or the effective potential, we conclude that the energy E of the total Hamiltonian H_{ξ} for bounded motions must satisfy the following inequalities depending on κ :

$$\begin{aligned} \kappa < 0, \quad & -\sqrt{|\kappa|} > E \geq -\frac{1}{2} \left(\frac{1}{\ell^2} + |\kappa|\ell^2 \right), \\ \kappa = 0, \quad & 0 > E \geq -\frac{1}{2\ell^2}, \\ \kappa > 0, \quad & \infty > E \geq -\frac{1}{2} \left(\frac{1}{\ell^2} - \kappa\ell^2 \right). \end{aligned}$$

- *Ladder functions of H_{θ}*

In order to find the ladder functions for the angular Hamiltonian H_{θ} , first we multiply it by $\sin^2 \theta$ and after rearranging, we get

$$-\ell_z^2 = p_{\theta}^2 \sin^2 \theta - H_{\theta} \sin^2 \theta.$$

Now, we can factorize the right hand side of this equality in terms of complex conjugate ladder functions

$$-\ell_z^2 = A^{+} A^{-} + \lambda_{\theta},$$

where

$$A^{\pm} = \mp i \sin \theta p_{\theta} + \sqrt{H_{\theta}} \cos \theta, \quad \lambda_{\theta} = -H_{\theta}.$$

These ladder functions A^{\pm} together with the Hamiltonian H_{θ} satisfy the following PBs

$$\{A^{-}, A^{+}\} = 2i \sqrt{H_{\theta}}, \quad \{H_{\theta}, A^{\pm}\} = \pm 2i \sqrt{H_{\theta}} A^{\pm}.$$

Therefore, the PB with H is

$$\{H, A^{\pm}\} = \mp i \frac{\sqrt{H_{\theta\varphi}}}{S_{\kappa}^2(\xi)} A^{\pm}.$$

Notice that the factorization implies the following inequality:

$$\ell^2 \geq \ell_z^2$$

which is reasonable in view of the physical interpretation of ℓ (the total angular momentum) and ℓ_z (the component of the angular momentum in the z -direction).

- *Construction of X^\pm*

From the PBs between B^+ and B^- , taking into account that $\beta = m/n$, we obtain

$$\{H, X^\pm\} = 0, \quad X^\pm = (A^\pm)^m (B^\mp)^n.$$

Hence

$$X^\pm = q_x e^{\pm i \alpha_x}, \quad 0 \leq q_x < \infty, \quad -\pi \leq \alpha_x < \pi.$$

The absolute value q_x is directly obtained from the factorization properties of A^\pm and B^\pm :

$$q_x = |X^\pm| = (\ell^2 - \ell_z^2)^{m/2} \left(E + \frac{1}{2} \left(\frac{1}{\ell^2} - \kappa \ell^2 \right) \right)^{n/2}.$$

By means of these constants of motion, one can easily find the relation between the variables ξ (or r) and θ along the trajectories, as well as the associate frequencies. However, when $\ell = \ell_z$, this relation breaks, because $p_\theta = 0$ and $\theta = \pi/2$. In other words, if $\ell = \ell_z$, the motion takes place in the horizontal plane $z = 0$. This particular case will be considered in a separate subsection.

Notice that X^\pm are not polynomial in momenta, since A^\pm and B^\pm depend on ℓ which is a square root. However, expanding $m - \hbar$ and $n - \hbar$ powers, and taking real and imaginary parts, one gets constants that are polynomial in momenta.

- (ii) *The constants of motion Y^\pm*

This new set of constants of motion is derived from the shift functions of H_θ and the ladder functions of H_φ . They are obtained in a similar way as the previous set.

- *Shift functions of H_θ*

The angular Hamiltonian is factorized as

$$H_\theta = p_\theta^2 + \frac{\ell_z^2}{\sin^2 \theta} = C^+ C^- + \lambda_\ell,$$

where

$$C^\pm = \mp i p_\theta + \ell_z \cot \theta, \quad \lambda_\ell = \ell_z^2.$$

These factor functions C^\pm together with the Hamiltonian H_θ satisfy the following PBs

$$\{C^-, C^+\} = 2i \frac{\ell_z}{\sin^2 \theta}, \quad \{H_\theta, C^\pm\} = \pm 2i \frac{\ell_z}{\sin^2 \theta} C^\pm.$$

The second PB implies that

$$\{H_{\theta\varphi}, C^\pm\} = \pm 2i \frac{\sqrt{H_\varphi}}{\sin^2 \theta} C^\pm.$$

- *Ladder functions of H_φ*

The Hamiltonian $H_\varphi = p_\varphi^2$ is factorized as

$$H_\varphi = D^+ D^-, \quad D^\pm = \sqrt{H_\varphi} e^{\mp i \varphi}.$$

These ladder functions D^\pm together with the Hamiltonian H_φ satisfy the following PBs

$$\{D^-, D^+\} = 2i \sqrt{H_\varphi}, \quad \{H_\varphi, D^\pm\} = \pm 2i \sqrt{H_\varphi} D^\pm.$$

Then, the PB with the Hamiltonian $H_{\theta\varphi}$ is

$$\{H_{\theta\varphi}, D^\pm\} = \pm 2i \frac{\sqrt{H_\varphi}}{\sin^2 \theta} D^\pm.$$

- *Final expressions for Y^\pm*

The Hamiltonian $H_{\theta\varphi}$ has the constants of motion Y^\pm .

$$\{H_{\theta\varphi}, Y^\pm\} = 0, \quad Y^\pm = (C^\pm)(D^\mp).$$

The geometric meaning of these constants of motion can be appreciated by rewriting them in the form

$$Y^\pm = -L_z(L_x \pm iL_y),$$

where L_x , L_y and L_z are the components of the angular momentum in the x , y and z direction, respectively:

$$L_x = -\sin \varphi p_\theta - \cot \theta \cos \varphi p_\varphi, \quad L_y = \cos \varphi p_\theta - \cot \theta \sin \varphi p_\varphi, \quad L_z = p_\varphi.$$

On the other hand they can be cast in the form

$$Y^\pm = q_y e^{\pm i\alpha_y}, \quad 0 \leq q_y < \infty, \quad \pi \leq \alpha_y < \pi,$$

with

$$q_y = |Y^\pm| = \ell_z(\ell^2 - \ell_z^2)^{1/2},$$

while the angle α_y coincides with the azimuthal angle of the angular momentum vector $\mathbf{L} = (L_x, L_y, L_z)$.

The value of this constant of motion fixes the relation of the angles θ and φ along a trajectory. For example, for $\ell_y = 0$, $\alpha_y = \pi$, one gets the following expression for $\theta(\varphi)$ (or $\varphi(\theta)$)

$$\cot \theta = -\frac{\ell_x}{\ell_z} \cos \varphi.$$

However, as mentioned before, this relation also breaks when $\ell = \ell_z$, since in this case $\ell_x = 0$ and therefore $\theta = \pi/2$.

It turns out that Y^\pm are polynomial in the momentum variables.

Thus, we have arrived at an obvious result: the Hamiltonian \mathbf{L}^2 has the symmetries L_z and Y^\pm or equivalently L_x , L_y and L_z . The basis L_z , Y^\pm will be the most adequate to write the symmetry algebra.

4. The Trajectories

From the results derived in the previous sections it follows that, once the values of the constants of motion E, ℓ, ℓ_z have been fixed, the new constants of motion X^\pm and Y^\pm determine the relation between the coordinates $r-\theta$ and $\theta-\varphi$, respectively. In this way, one can obtain all the trajectories of the system. In order to characterize those trajectories we will use the properties of the ladder and shift functions.

The ladder functions B^\pm , and the shift functions, A^\pm , can be expressed as

$$B^\pm(\xi, p_\xi) = \left(E + \frac{1}{2} \left(\frac{1}{\ell^2} - \kappa \ell^2\right)\right)^{1/2} e^{\pm i b(\xi, p_\xi)},$$

$$A^\pm(\theta, p_\theta) = (\ell^2 - \ell_z^2)^{1/2} e^{\pm i a(\theta, p_\theta)},$$

where $b(\xi, p_\xi)$ and $a(\theta, p_\theta)$ are real phase functions that depend also on the constants of motion E, ℓ and ℓ_z .

The effective Hamiltonian H_ξ depends of the variables (ξ, p_ξ) . When the energy E satisfies the proper restrictions the motion is periodic between two turning points. The variables (θ, p_θ)

are described by the effective Hamiltonian H_θ . For $\ell_z \neq 0$, the motion of these variables will be periodic, and the range of θ is determined by its corresponding turning points.

As a consequence, the functions $b(\xi, p_\xi)$ and $a(\theta, p_\theta)$ will also be periodic. Now, taking into account the constants of motion X^\pm , the phases of A^\pm , B^\pm and X^\pm are related as follows

$$m a(\theta, p_\theta) - n b(\xi, p_\xi) = \alpha_x, \quad \xi_1 \leq \xi \leq \xi_2, \quad \theta_1 \leq \theta \leq \theta_2.$$

This equation fixes the relation of the variables (ξ, p_ξ) and (θ, p_θ) along the motion. Therefore, if we time-differentiate the above relation between the phases, we get

$$m \dot{a}(\theta, p_\theta) - n \dot{b}(\xi, p_\xi) = 0.$$

This implies that the frequencies ω_ξ and ω_θ are related by

$$m \omega_\theta - n \omega_\xi = 0.$$

In the same way, we can write the functions C^\pm and D^\pm in the form

$$C^\pm(\theta, p_\theta) = (\ell^2 - \ell_z^2)^{1/2} e^{\pm i c(\theta, p_\theta)},$$

$$D^\pm(\varphi, p_\varphi) = \ell_z e^{\pm i d(\varphi, p_\varphi)},$$

where $c(\theta, p_\theta)$ and $d(\varphi, p_\varphi)$ are real phase functions. Due to the constants of motion Y^\pm these variables are related by

$$c(\theta, p_\theta) - d(\varphi, p_\varphi) = \alpha_Y, \quad \theta_1 \leq \theta \leq \theta_2, \quad -\pi \leq \varphi < \pi.$$

As the motion of the (θ, p_θ) variables and the (φ, p_φ) is periodic, differentiating the above equality with respect to time, we obtain

$$\omega_\theta - \omega_\varphi = 0.$$

Hence, the frequencies of the angular variables θ and φ are equal (except for the case $\ell = \ell_z$).

In conclusion, when the energy E satisfies the above restrictions, the motion is bounded, and the frequencies of the three variables are related as we have shown. This implies that the bounded motion is periodic and the trajectories are closed. In conclusion, we have checked in this case that the Bertrand's theorem is satisfied.

In Fig. 2 and Fig. 3 some examples of the trajectories for different values of the energies E and of the parameter β corresponding to bounded and unbounded motions are shown for the case $\kappa = -1$. The trajectories of the initial Hamiltonian are plotted in a three dimensional space $(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)$, where (r, φ, θ) are the spherical coordinates in \mathbb{R}^3 .

5. Algebra of the constants of motion for the Perlick system type I

As we have seen in the previous sections, we have obtained seven constants of motion for the three dimensional Perlick system: $H, H_{\theta\varphi}, H_\varphi, X^\pm, Y^\pm$. Of course, only five of these constants are independent. For example we can choose: $H, H_{\theta\varphi}, H_\varphi, X^+, Y^+$. Therefore, the Perlick system type I for a rational $\beta = m/n$ is a maximally superintegrable system. It turns out that, all the aforementioned seven constants are useful to express in a simple form the algebraic

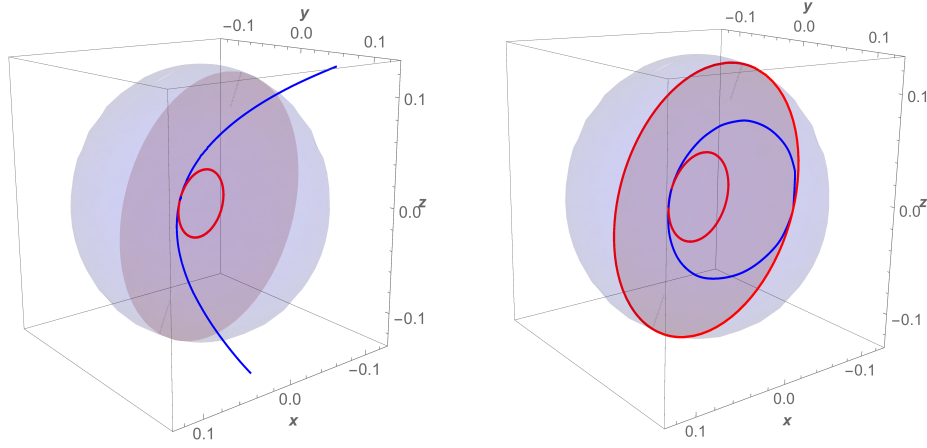


Figure 2. Plot of the trajectories of $\beta = 1$ for the values $E = -1$ (left), $E = -6$ (right), $\kappa = -1$, $\ell = 0.25$ and $\ell_z = 0.1$.

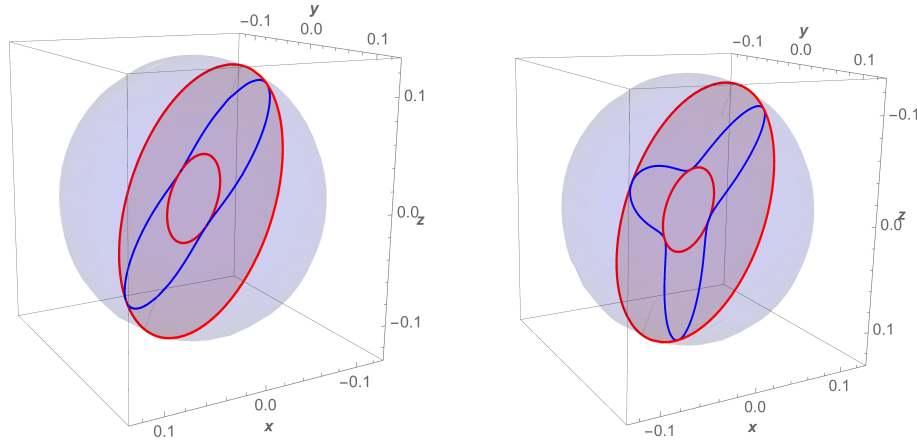


Figure 3. Plot of the trajectories of $\beta = 2$ (left), $\beta = 3$ (right) for the values $E = -6$, $\kappa = -1$, $\ell = 0.25$ and $\ell_z = 0.1$.

structure defined in terms of Poisson brackets:

$$\begin{aligned}
 \{H, \cdot\} &= 0, & \{H_{\theta\varphi}, Y^{\pm}\} &= 0, & \{H_{\theta\varphi}, X^{\pm}\} &= \pm 2im\sqrt{H_{\theta\varphi}}X^{\pm}, \\
 \{H_{\varphi}, Y^{\pm}\} &= \mp 2i\sqrt{H_{\varphi}}Y^{\pm}, & \{H_{\varphi}, X^{\pm}\} &= 0, & \{Y^+, Y^-\} &= 2i\sqrt{H_{\varphi}}(H_{\theta\varphi} - 2H_{\varphi}), \\
 \{X^+, X^-\} &= imn(H_{\theta\varphi} - H_{\varphi})^m \left(H_{\xi} + \frac{1}{2} \left(\frac{1}{H_{\theta\varphi}} - \kappa H_{\theta\varphi} \right)^{n-1} \left(\kappa\sqrt{H_{\theta\varphi}} + \frac{1}{H_{\theta\varphi}^{3/2}} \right) \right. \\
 &\quad \left. - 2im^2(H_{\theta\varphi} - H_{\varphi})^{m-1} \left(H_{\xi} + \frac{1}{2} \left(\frac{1}{H_{\theta\varphi}} - \kappa H_{\theta\varphi} \right)^n \sqrt{H_{\theta\varphi}} \right) \right), \\
 \{X^{\pm}, Y^{\pm}\} &= \mp \frac{im}{\sqrt{H_{\theta\varphi}} + \sqrt{H_{\varphi}}} X^{\pm} Y^{\pm}, & \{X^{\pm}, Y^{\mp}\} &= \mp \frac{im}{\sqrt{H_{\theta\varphi}} - \sqrt{H_{\varphi}}} X^{\pm} Y^{\mp}.
 \end{aligned}$$

In this way, we have found the algebra of the constants of motion for any two coprime integers m and n and any value of the constant κ . The corresponding polynomial algebras can also be found. .

The complex constants of motion X^{\pm} and Y^{\pm} allow to get real constants: $Y_R = \text{Re}(Y^{\pm})$, $Y_I =$

$\text{Im}(Y^\pm)$, $X_R = \text{Re}(X^\pm)$, $X_I = \text{Im}(X^\pm)$, which will close a real algebra. Besides, for $\beta = 1, \kappa = 0$, the complex constants of motion X^\pm can be expressed in terms of the angular momentum $\mathbf{L} = (L_x, L_y, L_z)$ and Runge-Lenz $\mathcal{A} = (\mathcal{A}_x, \mathcal{A}_y, \mathcal{A}_z)$ vectors,

$$X^\pm = \text{Re}(X^\pm) \pm i \text{Im}(X^\pm) = \frac{\mathcal{A}_z}{\sqrt{2}} \pm i \frac{(\mathbf{L} \times \mathcal{A})_z}{\sqrt{2} \ell},$$

where

$$\mathcal{A} = \mathbf{p} \times \mathbf{L} - \hat{\mathbf{r}}.$$

Here, we have used the expressions of $\mathbf{r}, \mathbf{p}, \mathbf{L}$ in spherical coordinates. Notice that in the context of the analysis of the Perlick system, a generalization of the Runge-Lenz vector has been proposed in [7]. In the next section we will consider the motion in the equatorial plane $\theta = \pi/2$.

6. Perlick I on the Equatorial Plane

By choosing the plane described by the initial position and velocity of the “planet” to be the Horizontal plane $z = 0$ (or the “equatorial” plane $\theta = \pi/2$) the expression of the relevant constants of motion and of the corresponding trajectories simplify dramatically.

Indeed, by taking $\theta = \pi/2$ so that $\mathbf{L} = (0, 0, L_z)$ ($\ell = \ell_z$). the corresponding Hamiltonian in the remaining r, φ variables is

$$\tilde{H}^\pm = \beta^2(1 + K r^2) \frac{p_r^2}{2} + \frac{p_\varphi^2}{2 r^2} \pm \frac{1}{r} \sqrt{1 + K r^2}.$$

Now, we have two trivial constants of motion: the total energy E and ℓ_z . Following the procedure outlined in the previous sections, letting $(r, p_r) \rightarrow (\xi, p_\xi)$, we get the Hamiltonian

$$H(\xi, \theta, \varphi) = \beta^2 \frac{p_\xi^2}{2} + \frac{p_\varphi^2}{2} \frac{1}{S_\kappa^2(\xi)} - \frac{1}{\kappa(\xi)},$$

with the same range of the variable ξ . In this case the effective Hamiltonian H_ξ is given by

$$H_\xi = \beta^2 \frac{p_\xi^2}{2} + \frac{\ell_z^2}{2} \frac{1}{S_\kappa^2(\xi)} - \frac{1}{T_\kappa(\xi)} = \beta^2 \frac{p_\xi^2}{2} + V_{\text{eff}}(\xi).$$

We have two additional constants of motion Z^\pm for that Hamiltonian:

$$\{H, Z^\pm\} = 0, \quad Z^\pm = (D^\pm)^m (B^\mp)^n,$$

where

$$B^\pm = \frac{1}{\sqrt{2}} \left(\mp i \beta p_\xi + \frac{\ell_z}{T_\kappa(\xi)} - \frac{1}{\ell_z} \right), \quad D^\pm = \sqrt{H_\varphi} e^{\mp i \varphi},$$

and

$$H_\xi = B^+ B^- + \lambda_\xi, \quad \lambda_\xi = -\frac{1}{2} \left(\frac{1}{\ell_z^2} - \kappa \ell_z^2 \right),$$

$$H_\varphi = p_\varphi^2 = D^+ D^-.$$

These constants of motion Z^\pm have complex values denoted by

$$Z^\pm = q_z e^{\pm i \varphi_z},$$

where $-\pi \leq \varphi_z < \pi$ and $0 \leq q_z < \infty$. The modulus q_z has a value determined by the other constants of motion E and ℓ_z ,

$$q_z = |Z^\pm| = \ell_z^m \left(E + \frac{1}{2} \left(\frac{1}{\ell_z^2} - \kappa, \ell_z^2 \right) \right)^{n/2}.$$

The functions B^\pm and D^\pm can be written as

$$B^\pm(\xi, p_\xi) = \left(E + \frac{1}{2} \left(\frac{1}{\ell_z^2} - \kappa, \ell_z^2 \right) \right)^{1/2} e^{\pm i b(\xi, p_\xi)},$$

$$D^\pm(\varphi, p_\varphi) = \ell_z e^{\pm i d(\varphi, p_\varphi)},$$

where $b(\xi, p_\xi)$ and $d(\varphi, p_\varphi)$ are the corresponding real phase functions. When the energy E satisfies the bound-state restriction, the bounded motion of the variables (ξ, p_ξ) is periodic. Besides, the motion of the variables (φ, p_φ) is also periodic, due to the angular character of φ . Then we get

$$m d(\varphi, p_\varphi) - n b(\xi, p_\xi) = \varphi_z, \quad \xi_1 \leq \xi \leq \xi_2, \quad -\pi \leq \varphi < \pi.$$

Whence, by a t -differentiation:

$$m \omega_\varphi - n \omega_\xi = 0.$$

The constant of motion Z^\pm reads:

$$(\ell_z e^{\mp i \varphi})^m \left(\mp i \frac{\beta}{\sqrt{2}} p_\xi + \frac{\ell_z}{\sqrt{2}} \frac{1}{T_\kappa(\xi)} - \frac{1}{\ell_z \sqrt{2}} \right)^n = q_z e^{\pm i \varphi_z}.$$

Taking the real and imaginary parts of the equation above we obtain the relation between ξ and φ along the trajectories:

$$\cos \left(\frac{1}{n} \varphi_z + \frac{m}{n} \varphi \right) = \frac{\frac{\ell_z^2}{T_\kappa(\xi)} - 1}{\sqrt{2 E \ell_z^2 + 1 - \kappa \ell_z^4}}.$$

which becomes the well known conic section equation for the values $\kappa = 0$, $\beta = m/n = 1$ and $\varphi_z = 0$:

$$\frac{\alpha}{\xi} = 1 + \varepsilon \cos \varphi,$$

where $\varepsilon^2 = 2 E \ell_z^2 + 1$ (eccentricity) and $\alpha = \ell_z^2$ (semi-latus rectum) [10]. In this case, the problem is reduced to the Kepler-Coulomb system in the Euclidean plane. If $\kappa = -1$ ($\kappa = 1$) and $\beta = 1$, then we get the conic section equation for the hyperboloid (sphere) [11]

$$\kappa = -1, \quad (\text{Hyperbolic}) \quad \frac{\alpha}{\tanh \xi} = 1 + \sqrt{\varepsilon^2 + \alpha^2} \cos \varphi,$$

$$\kappa = 1, \quad (\text{Spherical}) \quad \frac{\alpha}{\tan \xi} = 1 + \sqrt{\varepsilon^2 - \alpha^2} \cos \varphi.$$

So, we may say that our trajectory equation can be considered as a generalized conic section equation.

In the special case where $\kappa = 0$ and $\beta = m/n = 1$, the constants of motion Z^\pm can also be expressed in terms of the Runge-Lenz vector $\mathcal{A} = (\mathcal{A}_x, \mathcal{A}_y, 0)$,

$$Z^\pm = \text{Re}(Z^\pm) \pm i \text{Im}(Z^\pm) = \frac{\mathcal{A}_x}{\sqrt{2}} \mp i \frac{\mathcal{A}_y}{\sqrt{2}}.$$

For the other cases ($\kappa = \pm 1$), the constants of motion Z^\pm can be written in terms of a type of generalized Runge-Lenz vector [7] Note that the constants H, H_φ, Z^\pm close a similar algebra as in the general case.

7. Concluding Remarks and Future Perspectives

In the above pages we have tersely recalled the algebraic and geometrical properties of the classical Perlick system of type I, which have been described in term of the factorization approach. The further challenging task is performing a similar construction for the full Perlick family II, on which some preliminary results (not yet published) have been already derived.

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