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THE EIGENVECTORS OF A COMBINATORIAL MATRIX

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Abstract. In this paper, we derive the eigenvectors of a combinatorial matrix whose eigenvalues studied by Kilic and Stanica. We follow the method of Cooper and Melham since they considered the special case of this matrix.

1. Introduction

In [7], Peele and Stănică studied $n \times n$ matrices with the (i,j) entry the binomial coefficient $\binom{i-1}{j-1}$, respectively, $\binom{i-1}{n-j}$ and derived many interesting results on powers of these matrices. In [8], one of them found that the same is true for a much larger class of what he called *netted matrices*, namely matrices with entries satisfying a certain type of recurrence among the entries of all 2×2 cells.

Let R_n be the matrix whose (i,j) entries are $a_{i,j} = \binom{i-1}{n-j}$, which satisfy

$$a_{i,j-1} = a_{i-1,j-1} + a_{i-1,j}. \tag{1.1}$$

The previous recurrence can be extended for $i \geq 0,\ j \geq 0$, using the boundary conditions $a_{1,n} = 1$, $a_{1,j} = 0$, $j \neq n$. Remark the following consequences of the boundary conditions and recurrence (1.1): $a_{i,j} = 0$ for $i + j \leq n$, and $a_{i,n+1} = 0, 1 \leq i \leq n$.

The matrix R_n was firstly studied by Carlitz [2] who gave explicit forms for the eigenvalues of R_n. Let $f_{n+1}(x) = \det(xI - R_n)$ be the characteristic polynomial of R_n. Thus

$$f_n(x) = \sum_{r=0}^{n+1} (-1)^{r(r+1)/2} \binom{n}{r}_F x^{n-r}$$

where $\binom{n}{r}_F$ denote the Fibonomial coefficient, defined (for $n \geq r > 0$) by

$$\binom{n}{r}_F = \frac{F_1 F_2 \ldots F_n}{(F_1 F_2 \ldots F_r)(F_1 F_2 \ldots F_{n-r})}$$

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with $\binom{n}{n}_F = \binom{n}{0}_F = 1$. Carlitz showed that

$$f_n(x) = \prod_{j=0}^{n-1} \left(x - \phi^j \bar{\phi}^{n-j} \right)$$

where $\phi, \bar{\phi} = \left(1 \pm \sqrt{5}\right)/2$. Thus the eigenvalues of R_n are $\phi^n, \phi^{n-1}\bar{\phi}, \ldots, \phi\bar{\phi}^{n-1}, \bar{\phi}^n$.

In [7] it was proved that the entries of the power R_n^e satisfy the recurrence

$$F_{e-1} a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1} a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)}, \tag{1.2}$$

where F_e is the Fibonacci sequence. Closed forms for *all* entries of R_n^e were not found, but several results concerning the generating functions of rows and columns were obtained (see [7, 8]). Further, the generating function for the (i,j)-th entry of the e-th power of a generalization of R_n, namely

$$Q_n(a,b) = \left(a^{i+j-n-1} b^{n-j} \binom{i-1}{n-j} \right)_{1 \le i,j \le n}$$

is

$$B_n^{(e)}(x,y) = \frac{(U_{e-1} + U_e y)(b\,U_{e-1} + y\,U_e)^{n-1}}{U_{e-1} + U_e y - x(U_e + U_{e+1} y)}.$$

Regarding this generalization, in [3], the authors gave the characteristic polynomial of $Q_n(a,b)$ and the trace of kth power of $Q_n(a,b,)$, that is, $\mathrm{tr}\big(Q_n^k(a,b)\big)$, by using the method of Carlitz [2]:

$$\mathrm{tr}\left(Q_n^k(a,b)\right) = \frac{U_{kn}}{U_k}, \tag{1.3}$$

where $\binom{n}{i}_U$ stands for the generalized Fibonomial coefficient, defined by

$$\binom{n}{r}_U = \frac{U_1 U_2 \ldots U_n}{(U_1 U_2 \ldots U_r)(U_1 U_2 \ldots U_{n-r})},$$

for $n \ge i > 0$, where $\binom{n}{n}_U = \binom{n}{0}_U = 1$.

In [7], the authors proposed a conjecture on the eigenvalues of matrix R_n, which was proven independently in [1] and the unpublished manuscript [9]. Also, in [6], they found the eigenvectors of R_n.

In [7, 8], it was shown that the inverse of R_n is the matrix

$$R_n^{-1} = \left((-1)^{n+i+j+1} \binom{n-i}{j-1} \right)_{1 \le i,j \le n},$$

and, in general, the inverse of $Q_n(a,b)$ is

$$Q_n^{-1}(a,b) = \left((-1)^{n+i+j+1} a^{n+1-i-j} b^{i-n} \binom{n-i}{j-1} \right)_{1 \le i,j \le n}. \tag{1.4}$$

Let $\phi = \frac{1+\sqrt{5}}{2}$, $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ be the golden section and its conjugate. The eigenvalues R_n are $\phi^n, \phi^{n-1}\bar{\phi}, \ldots, \phi\bar{\phi}^{n-1}, \bar{\phi}^n$.

Let the sequences $\{u_n\}$, $\{v_n\}$ be defined by

$$\begin{aligned} u_n &= au_{n-1} + bu_{n-2} \\ v_n &= av_{n-1} + bv_{n-2}, \end{aligned}$$

for $n > 1$, where $u_0 = 0, u_1 = 1$, and $v_0 = 2, v_1 = a$, respectively. Let α, β be the roots of the associated equation $x^2 - ax - b = 0$. The next lemma can be found in [4].

Lemma 1.1. *For $k \geq 1$ and $n > 1$,*

$$\begin{aligned} u_{kn} &= v_k u_{k(n-1)} + (-1)^{k+1} b^k u_{k(n-2)} \\ v_{kn} &= v_k v_{k(n-1)} + (-1)^{k+1} b^k v_{k(n-2)}. \end{aligned} \tag{1.5}$$

In [5], using the sequence v_k, they defined the $n \times n$ matrix $H_n\left(v_k, b^k\right)$ as follows:

$$H_n\left(v_k, b^k\right) = \left(v_k^{i+j-n-1} \left(-(-b)^k\right)^{n-j} \binom{i-1}{n-j} \right)_{1 \leq i,j \leq n}.$$

As in equation (1.4), they also found the inverse of the matrix H_n, namely

$$H_n^{-1}(v_k, b^k) = \left((-1)^{j+1} (-b)^{-k(n-i)} v^{n+1-i-j} \begin{pmatrix} n-i \\ j-1 \end{pmatrix} \right)_{i,j}.$$

It is well known that for $n \geq -1$,

$$u_{n+1} = \sum_r \binom{r}{n-r} a^{2r-n} b^{n-r}. \tag{1.6}$$

Thus the authors [5] generalized this identity as well as they gave the following results:

Lemma 1.2. *For $k > 0$ and $n \geq -1$,*

$$\frac{u_{k(n+1)}}{u_k} = \sum_r \binom{r}{n-r} v_k^{2r-n} \left(-(-b)^k\right)^{n-r}.$$

Lemma 1.3. *For all $m > 0$,*

$$tr\left(H_n^m\left(v_k, b^k\right)\right) = \frac{u_{knm}}{u_k}.$$

Theorem 1.4. *The eigenvalues of $H_n\left(v_k, b^k\right)$ are*

$$\alpha^{kn}, \alpha^{k(n-1)}\beta^k, \ldots, \alpha^k \beta^{k(n-1)}, \beta^{kn}.$$

2. The eigenvectors of $H_n\left(v_k, b^k\right)$

In [6], the authors considered the matrix $Q_n(a,b)$ and gave its eigenvectors. In this section, we consider the generalization of matrix $Q_n(a,b)$ namely $H_n\left(v_k, b^k\right)$ and then determine its eigenvectors by using the method given in [6].

Let $0 \le p \le n-1$ be a fixed integer,

$$f(x) = (x-\alpha^k)^p (x-\beta^k)^{n-1-p} = \sum_{r=0}^{n-1} s_r x^r,$$

and

$$\mathbf{s} = (s_0, s_1, ..., s_{n-1})^T.$$

Theorem 2.1. *For $m \ge 0$*

$$f^{(m)}(x) = m! \frac{f(x)}{(x-\alpha^k)^m (x-\beta^k)^m} \sum_{j=0}^{m} \binom{p}{m-j} \binom{n-1-p}{j} (x-\alpha^k)^j (x-\beta^k)^{m-j}.$$

Proof. It can be proved with the use of Leibniz's formula for the m-th derivative of a product of two functions. We recall the Leibniz's formula: For $m \ge 0$

$$\frac{d^m}{dx^m} g(x)h(x) = \sum_{j=0}^{m} \binom{m}{j} g^{(m-j)}(x) h^{(j)}(x). \tag{2.1}$$

We use the notation $x^{\underline{n}}$ to denote the falling factorial, and hence

$$\begin{aligned} f^{(m)}(x) &= \sum_{j=0}^{m} \binom{m}{j} p^{\underline{m-j}} (x-\alpha^k)^{p-m+j} (n-1-p)^{\underline{j}} (x-\beta^k)^{n-1-p-j} \\ &= m! \frac{f(x)}{(x-\alpha^k)^m (x-\beta^k)^m} \sum_{j=0}^{m} \binom{p}{m-j} \binom{n-1-p}{j} (x-\alpha^k)^j (x-\beta^k)^{m-j}, \end{aligned}$$

as claimed. □

Lemma 2.2. *Suppose that $0 \le m \le n-1$ be a fixed integer. Then,*

$$s_{n-1-m} = \sum_{j=0}^{m} (-1)^m \binom{p}{m-j} \binom{n-1-p}{j} \alpha^{k(m-j)} \beta^{kj}$$

and

$$\left(H_n\left(v_k, b^k\right) \mathbf{s}\right)_{n-1-m} = \sum_{r=m}^{n-1} (-1)^m (-b)^{km} \binom{r}{m} v_k^{r-m} s_r.$$

Proof. The proof of the first equation can be followed from computing the coefficient of x^{n-1-m} of $f(x)$ by multiplying $\left(x-\alpha^k\right)^p$ times $\left(x-\beta^k\right)^{n-1-p}$. The second proof can be seen by computing the product of $H_n\left(v_k, b^k\right)$ and $\mathbf{s}$. □

Theorem 2.3.
$$H_n\left(v_k, b^k\right)\mathbf{s} = (\alpha^k)^{n-1-p}\beta^{kp}\mathbf{s}. \qquad (2.2)$$

Proof. Consider

$$\begin{aligned}
\left(H_n\left(v_k, b^k\right)\mathbf{s}\right)_{n-1-m} &= \sum_{r=m}^{n-1}(-1)^m(-b)^{km}\binom{r}{m}v_k^{r-m}s_r \\
&= \frac{(-1)^m(-b)^{km}}{m!}\sum_{r=m}^{n-1}s_r r^{\underline{m}}v_k^{r-m} \\
&= \frac{(-1)^m(-b)^{km}}{m!}f^{(m)}(v_k) \\
&= \frac{(-1)^m(-b)^{km}}{m!}\frac{(v_k-\alpha^k)^p(v_k-\beta^k)^{n-1-p}}{(v_k-\alpha^k)^m(v_k-\beta^k)^m}\times \\
&\quad m!\sum_{j=0}^{m}\binom{p}{m-j}\binom{n-1-p}{j}(v_k-\alpha^k)^j(v_k-\beta^k)^{m-j} \\
&= (-1)^m(\alpha\beta)^{km}\beta^{k(p-m)}\alpha^{k(n-1-p-m)}\times \\
&\quad \sum_{j=0}^{m}\binom{p}{m-j}\binom{n-1-p}{j}\beta^{kj}\alpha^{k(m-j)} \\
&= \alpha^{k(n-1-p)}\beta^{kp}\sum_{j=0}^{m}(-1)^m\binom{p}{m-j}\binom{n-1-p}{j}\alpha^{k(m-j)}\beta^{kj} \\
&= \alpha^{k(n-1-p)}\beta^{kp}s_{n-1-m}.
\end{aligned}$$

Thus the proof is complete. □

ÖZET: Bu çalışmada, Kilic ve Stanica tarafından özdeğerleri verilen bir kombinatoryal matrisin özvektörleri, Cooper ve Melham'ın metodu takip edilerek elde edilmiştir.

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