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OSCULATING SPHERES AND OSCULATING CIRCLES OF A CURVE IN SEMI-RIEMANNIAN SPACE

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ABSTRACT. In the Euclidean 3-space, there is a unique sphere for a curve $\alpha : I \to \mathbb{R}^3$ such that the sphere touches α at the third order at $\alpha(0)$. The intersection of the sphere with osculating plane is a circle which touches α at the second order at $\alpha(0)$ [5]. In this paper, the osculating sphere and the osculating circle of the curve are studied for each of timelike, spacelike and null (lightlike) curves in Semi-Riemannian Spaces; $\mathbb{R}^3_1$, $\mathbb{R}^4_1$ and $\mathbb{R}^4_2$.

1. INTRODUCTION

The Semi-Riemannian n-space $\mathbb{R}^n_\nu$, is the Euclidean n-space $\mathbb{R}^n$ with the Semi-Riemannian inner product

$$\langle X,Y\rangle = -\sum_{i=1}^{\nu} x_i y_i + \sum_{j=\nu+1}^{n} x_j y_j \tag{1.1}$$

where $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$.

An arbitrary vector $X = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n_\nu$ can have one of the three causal characters; it is spacelike if $\langle X, X\rangle > 0$ or $X = 0$, timelike if $\langle X, X\rangle < 0$, $X \neq 0$ and null (lightlike) if $\langle X, X\rangle = 0$, $X \neq 0$. Similarly, an arbitrary curve $\alpha : I \to \mathbb{R}^n_\nu$, $s \to \alpha(s)$ in $\mathbb{R}^n_\nu$, where s is a pseudo-arclength parameter, can locally be spacelike, timelike or null, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null for every $s \in I$.

A *pseudosphere* of radius $r > 0$ in $\mathbb{R}^n_\nu$ is the hyperquadric

$$S^{n-1}_\nu(r) = \left\{p \in \mathbb{R}^n_\nu : \langle p,p\rangle = r^2\right\}.$$

Similarly, a *pseudohyperbolic space* of radius $r > 0$ in $\mathbb{R}^n_\nu$ is the hyperquadric

$$H^{n-1}_{\nu-1}(r) = \left\{p \in \mathbb{R}^n_\nu : \langle p,p\rangle = -r^2\right\}.$$

Let $\{T, N, B_1, B_2\}$ the moving Frenet frame along the curve α. Here, T is the tangent vector field, N is the principal normal vector field, B_1 and B_2 are the first

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and second binormal vector fields of the curve α. Depending on the causal character of the curve α, we have the following Frenet formulae in $\mathbb{R}^3_1, \mathbb{R}^4_1$ and $\mathbb{R}^4_2$.

1.1. **Frenet Frame in R^3_1.**

Case 1: If α is a spacelike curve with a timelike principal normal N,

$T' = k_1 N,\ N' = k_1 T + k_2 B,\ B' = k_2 N,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B,B\rangle = 1$, $\langle T,N\rangle = 0$, $\langle T,B\rangle = 0$ and $\langle N,B\rangle = 0$.

If α is a spacelike curve with a spacelike principal normal N,

$T' = k_1 N,\ N' = -k_1 T + k_2 B,\ B' = k_2 N,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B,B\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B\rangle = 0$, $\langle N,B\rangle = 0$.

Case 2: If α is a timelike curve,

$T' = k_1 N,\ N' = k_1 T + k_2 B,\ B' = -k_2 N,$

where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B,B\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B\rangle = 0$, $\langle N,B\rangle = 0$.

Case 3: If α is a null curve,

$T' = k_1 B,\ N' = -k_2 B,\ B' = -k_2 T + k_1 N,$

where $\langle T,T\rangle = 0$, $\langle T,N\rangle = 1$, $\langle T,B\rangle = 0$, $\langle N,N\rangle = 0$, $\langle N,B\rangle = 0$, $\langle B,B\rangle = 1$.

1.2. **Frenet Frame in R^4_1.**

Case 1: If α is a spacelike curve with a timelike principal normal N,

$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = k_2 N + k_3 B_2$ and $B_2' = -k_3 B_1,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B_1,B_1\rangle = 1, \langle B_2,B_2\rangle = 1$, $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0$, $\langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a spacelike curve with a timelike first binormal B_1,

$T' = k_1 N,\ N' = -k_1 T + k_2 B_1,\ B_1' = k_2 N + k_3 B_2$ and $B_2' = k_3 B_1,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = -1, \langle B_2,B_2\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a spacelike curve with a timelike second binormal B_2,

$T' = k_1 N,\ N' = -k_1 T + k_2 B_1,\ B_1' = -k_2 N + k_3 B_2$ and $B_2' = k_3 B_1,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = -1, \langle B_2,B_2\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0$, $\langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

Case 2: If α is a timelike curve,

$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = -k_2 N + k_3 B_2,\ B_2' = -k_3 B_1,$

where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = 1$, $\langle B_2,B_2\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$ $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

Case 3: If α is a null curve,

$T' = k_1 B_1,\ N' = -k_2 B_1 - k_3 B_2,\ B_1' = -k_2 T + k_1 N$ and $B_2' = -k_3 T,$

where $\langle T,T\rangle = 0$, $\langle T,N\rangle = -1$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,N\rangle = 0$, $\langle N,B_1\rangle = 0$, $\langle N,B_2\rangle = 0$, $\langle B_1,B_1\rangle = 1$ and $\langle B_2,B_2\rangle = 1$.

1.3. **Frenet Frame in R^4_2.**

Case 1: If α is a spacelike curve with timelike principal normal N and first timelike binormal B_1,

$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = -k_2 N + k_3 B_2$ and $B_2' = k_3 B_1,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B_1,B_1\rangle = -1, \langle B_2,B_2\rangle = 1$, $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0$, $\langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a spacelike curve with a timelike principal N and second binormal B_2,
$T' = k_1N,\ N' = k_1T + k_2B_1,\ B_1' = k_2N + k_3B_2$ and $B_2' = k_3B_1$,
where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B_1,B_1\rangle = 1, \langle B_2,B_2\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a spacelike curve with a timelike first binormal B_1 and second normal B_2,
$T' = k_1N,\ N' = -k_1T + k_2B_1,\ B_1' = k_2N + k_3B_2$ and $B_2' = -k_3B_1$,
where $\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = 1, \langle B_2,B_2\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

Case 2: If α is a timelike curve with timelike principal normal N,
$T' = k_1N,\ N' = -k_1T + k_2B_1,\ B_1' = k_2N + k_3B_2,\ B_2' = -k_3B_1$,
where $\langle T,T\rangle = -1$, $\langle N,N\rangle = -1$, $\langle B_1,B_1\rangle = 1$, $\langle B_2,B_2\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$ $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a timelike curve with timelike first binormal B_1,
$T' = k_1N,\ N' = k_1T + k_2B_1,\ B_1' = k_2N + k_3B_2,\ B_2' = k_3B_1$,
where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = -1$, $\langle B_2,B_2\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$ $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a timelike curve with timelike second binormal B_2,
$T' = k_1N,\ N' = k_1T + k_2B_1,\ B_1' = -k_2N + k_3B_2,\ B_2' = k_3B_1$,
where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = 1$, $\langle B_2,B_2\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$ $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

Case 3: If α is a null curve with timelike first binormal B_1,
$T' = -k_1B_1,\ N' = k_2B_1 - k_3B_2,\ B_1' = -k_2T + k_1N$ and $B_2' = -k_3T$,
where $\langle T,T\rangle = 0$, $\langle T,N\rangle = -1$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,N\rangle = 0$, $\langle N,B_1\rangle = 0$, $\langle N,B_2\rangle = 0$, $\langle B_1,B_1\rangle = -1$ and $\langle B_2,B_2\rangle = 1$.

If α is a null curve with timelike second binormal B_2,
$T' = k_1B_1,\ N' = -k_2B_1 + k_3B_2,\ B_1' = -k_2T + k_1N$ and $B_2' = -k_3T$,
where $\langle T,T\rangle = 0$, $\langle T,N\rangle = -1$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,N\rangle = 0$, $\langle N,B_1\rangle = 0$, $\langle N,B_2\rangle = 0$, $\langle B_1,B_1\rangle = 1$ and $\langle B_2,B_2\rangle = -1$. [1], [2], [3] and [4].

2. Osculating Sphere of a Timelike Curve

We shall assume that the timelike curve $\alpha : I \to \mathbb{R}^3_1$ is parametrized such that $||\alpha'(t)|| = 1$. Then we have $\alpha'(t) = T$. Let (y_1, y_2, y_3) be the Euclidean coordinate system in $\mathbb{R}^3_1$. We take a sphere $\langle y - d, y - d\rangle = r^2$, with origin and radius d and r respectively, where $y = (y_1, y_2, y_3)$. Let $f(t) = \langle \alpha(t) - d, \alpha(t) - d\rangle - r^2$. If we have the following equations

$$f(0) = 0,\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = 0 \tag{2.1}$$

then we say that the sphere touches α at the third order to the curve at $\alpha(0)$.

Theorem 2.1. *Let $k_1(0)$ and $k_2(0)$, the curvatures of a timelike curve $\alpha : I \to \mathbb{R}^3_1$ at $\alpha(0)$, be different from zero. Then there exists a sphere which touches at the third order to the curve at $\alpha(0)$, and the equation of the sphere according to the frame $\{T_0, N_0, B_0\}$ is*

$$-x_1^2 + (x_2 + \rho_o)^2 + (x_3 + \rho_o'\sigma_o)^2 = \rho_o^2 + (\rho_o'\sigma_o)^2, \tag{2.2}$$

where $\rho_o = \dfrac{1}{k_1(0)}$ and $\sigma_o = \dfrac{1}{k_2(0)}$.

Proof. If $f(0)=0$ then $\langle \alpha(0)-d, \alpha(0)-d\rangle = r^2$. Since we have $f' = 2\langle \alpha', \alpha - d\rangle$, $f'(0)=0$ implies $\langle T_o, \alpha(0)-d\rangle = 0$. Similarly we have $f'' = 2\left[\langle \alpha'', \alpha-d\rangle + \langle \alpha', \alpha'\rangle\right]$, $f''(0)=0$ implies $\langle k_1(0)N_o, \alpha(0)-d\rangle + \langle T_o, T_o\rangle = 0$. Substituting, $\langle T_o, T_o\rangle = -1$ in this equation we obtain $\langle N_o, \alpha(0)-d\rangle = \rho_o$ is obtained.

Considering $f''' = 2\left[\langle \alpha''', \alpha-d\rangle + 3\langle \alpha'', \alpha'\rangle\right]$ and $f'''(0)=0$ we get
$\langle k_1^2(0)T_o + k_1'(0)N_o + k_1(0)k_2(0)B_o, \alpha(0)-d\rangle + 3\langle k_1(0)N_o, T_o\rangle = 0.$
Consequently,

$$k_1^2(0)\langle T_o, \alpha(0)-d\rangle + k_1'(0)\langle N_o, \alpha(0)-d\rangle$$
$$+k_1(0)k_2(0)\langle B_o, \alpha(0)-d\rangle + 3k_1(0)\langle N_o, T_o\rangle = 0.$$

Here, Substituting $\langle T_o, \alpha(0)-d\rangle = 0$, $\langle N_o, \alpha(0)-d\rangle = \rho_o$, $\langle N_o, T_o\rangle = 0$, we obtain

$$\langle B_o, \alpha(0)-d\rangle = \frac{-\rho_o k_1'(0)}{k_1(0)k_2(0)} = \frac{-k_1'(0)}{k_1^2(0)k_2(0)} = \rho_o'\sigma_o.$$

Now we investigate the numbers u_1, u_2, u_3 such that $\alpha(0)-d = u_1T_o + u_2N_o + u_3B_o$. Since $\langle T_o, \alpha(0)-d\rangle = -u_1$ and $\langle T_o, \alpha(0)-d\rangle = 0$, then we find $u_1 = 0$. Since $\langle N_o, \alpha(0)-d\rangle = u_2$ and $\langle N_o, \alpha(0)-d\rangle = \rho_o$ then we find $u_2 = \rho_o$. Since $\langle B_o, \alpha(0)-d\rangle = u_3$ and $\langle B_o, \alpha(0)-d\rangle = \rho_o'\sigma_o$, then we find $u_3 = \rho_o'\sigma_o$. Also, the origin of the sphere that contacts at the third order to the curve at the point $\alpha(0)$ is

$$d = \alpha(0) - \rho_o N_o - \rho_o'\sigma_o B_o. \tag{2.3}$$

Given a variable P on the osculating sphere, suppose $P = \alpha(0) + x_1T_o + x_2N_o + x_3B_o$. Hence,

$$P - d = x_1T_o + (x_2+\rho_o)N_o + (x_3 + \rho_o'\sigma_o)B_o$$

also

$$\langle P-d,\ P-d\rangle = -x_1^2 + (x_2+\rho_o)^2 + (x_3+\rho_o'\sigma_o)^2$$

using (2.3), we obtain

$$r^2 = \langle \alpha(0)-d, \alpha(0)-d\rangle = \rho_o^2 + (\rho_o'\sigma_o)^2.$$

□

Now, we show that the circle which is the intersection of the osculating sphere at $\alpha(0)$ with the plane $Sp\{T_o, N_o\}$, contacts at the second order to the curve at $\alpha(0)$. This circle is called osculating circle of the curve at $\alpha(0)$.

Theorem 2.2. *For each timelike curve $\alpha : I \to \mathbb{R}_1^3$, there exist a curve $\gamma : \mathbb{R} \to \mathbb{R}_1^3$,*

$$\gamma(\theta) = \alpha(0) + (\rho_o \sinh\theta)T_o + \rho_o(-1+\cosh\theta)N_o \tag{2.4}$$

which contacts α at the second order at $\alpha(0)$.

Proof. The equation of the intersection of the plane $Sp\{T_o, N_o\}$ with the sphere which is given by (2.2) according to the frame $\{T_o, N_o, B_o\}$ is,

$$-x_1^2 + (x_2+\rho_o)^2 = \rho_o^2.$$

From this equation we can write $x_1 = \rho_o \sinh\theta$, $x_2 = \rho_o(-1+\cosh\theta)$. Thus the intersection circle can be given by (2.4). Clearly $\gamma(0) = \alpha(0)$.
Since $\gamma'(\theta) = (\rho_o\cosh\theta)T_o + (\rho_o\sinh\theta)N_o$, we have $\gamma'(0) = \rho_oT_o = \rho_o\alpha'(0)$. Also, $\gamma''(\theta) = (\rho_o\sinh\theta)T_o + (\rho_o\cosh\theta)N_o$ implies $\gamma''(0) = \rho_oN_o = \rho_o^2\alpha''(0)$.

The equalities $\gamma(0) = \alpha(0)$, $\gamma'(0) = \rho_o\alpha'(0)$ and $\gamma''(0) = \rho_o^2\alpha''(0)$ show that the curve γ touches α at the second order at $\alpha(0)$. □

Corollary 1. *Osculating circle of a timelike curve $\alpha : I \to \mathbb{R}_1^3$ at $\alpha(0)$ is also a timelike curve.*

Proof. It is easy to see that for every $\theta \in R$, $\langle\gamma'(\theta), \gamma'(\theta)\rangle = -\rho_o^2(\cosh\theta)^2 + \rho_o^2(\sinh\theta)^2 = -\rho_o^2 < 0$. □

We can state the following theorems for the osculating sphere of a timelike curve in $\mathbb{R}_1^4$ and $\mathbb{R}_2^4$ as follows:

Theorem 2.3. *Let $k_1(0)$, $k_2(0)$ and $k_3(0)$, the curvatures of a timelike curve $\alpha : I \to \mathbb{R}_1^4$ at $\alpha(0)$, be different from zero. Then there exist a sphere which touches at the fourth order to at $\alpha(0)$ and equation of the sphere according to the frame $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ is*

$$-x_1^2 + (x_2 + \lambda_1)^2 + (x_3 + \lambda_2)^2 + (x_4 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.5}$$

where $\lambda_1 = \rho_o$, $\lambda_2 = \rho_o'\sigma_o$, $\lambda_3 = \left((\rho_o'\sigma_o)' + \frac{\rho_o}{\sigma_o}\right)\omega_0$ and $\rho_o = \dfrac{1}{k_1(0)}$, $\sigma_o = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 2.4. *Let $k_1(0)$, $k_2(0)$ and $k_3(0)$, the curvatures of a timelike curve $\alpha : I \to \mathbb{R}_2^4$ at $\alpha(0)$ with timelike principal normal N, be different from zero. Then there exist a sphere which touches at the fourth order to α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ is*

$$-x_1^2 - (x_2 - \lambda_1)^2 + (x_3 + \lambda_2)^2 + (x_4 + \lambda_3)^2 = -\lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.6}$$

where $\lambda_1 = \rho_o$, $\lambda_2 = \rho_o'\sigma_o$, $\lambda_3 = \left((\rho_o'\sigma_o)' - \frac{\rho_o}{\sigma_o}\right)\omega_0$ and $\rho_o = \dfrac{1}{k_1(0)}$, $\sigma_o = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 2.5. *Let $k_1(0)$, $k_2(0)$ and $k_3(0)$,the curvatures of a timelike curve $\alpha : I \to \mathbb{R}_2^4$ at $\alpha(0)$ with timelike first binormal B_1, be different from zero. Then there exist a sphere which touches at the fourth order to α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ is*

$$-x_1^2 + (x_2 + \lambda_1)^2 - (x_3 + \lambda_2)^2 + (x_4 + \lambda_3)^2 = \lambda_1^2 - \lambda_2^2 + \lambda_3^2, \tag{2.7}$$

where $\lambda_1 = \rho_o$, $\lambda_2 = \rho_o'\sigma_o$, $\lambda_3 = \left((\rho_o'\sigma_o)' - \frac{\rho_o}{\sigma_o}\right)\omega_0$ and $\rho_o = \dfrac{1}{k_1(0)}$, $\sigma_o = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 2.6. *Let $k_1(0)$, $k_2(0)$ and $k_3(0)$, the curvatures of a timelike curve $\alpha : I \to \mathbb{R}_2^4$ at $\alpha(0)$ with timelike second binormal B_2, be different from zero. Then there exist a sphere which touches fourth order to α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ is*

$$-x_1^2 + (x_2 + \lambda_1)^2 + (x_3 - \lambda_2)^2 - (x_4 - \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 - \lambda_3^2, \tag{2.8}$$

where $\lambda_1 = \rho_o$, $\lambda_2 = \rho_o'\sigma_o$, $\lambda_3 = \left((\rho_o'\sigma_o)' + \frac{\rho_o}{\sigma_o}\right)\omega_0$ and $\rho_o = \dfrac{1}{k_1(0)}$, $\sigma_o = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

3. Osculating Sphere of a Null Curve

Theorem 3.1. *Let $k_1(0) \neq 0$. The sphere that contacts at the third order to null (lightlike) curve $\alpha : I \to \mathbb{R}_1^3$ at $\alpha(0)$ is the pseudosphere at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_0\}$ is*

$$-2x_1x_2 + x_3^2 = 0. \tag{3.1}$$

Proof. If $f(0) = 0$, then $\langle \alpha(0) - d, \alpha(0) - d\rangle = r^2$. Since $f' = 2\langle \alpha', \alpha - d\rangle$ $f'(0) = 0$ implies $\langle T_o, \alpha(0) - d\rangle = 0$. Since $f'' = 2k_1\langle B, \alpha - d\rangle$, $f''(0) = 0$ implies

$2k_1(0)\langle B_o, \alpha(0) - d\rangle = 0$. Since $k_1(0) \neq 0$ then we have $\langle B_o, \alpha(0) - d\rangle = 0$. Since

$$f''' = 2[k_1'\langle B, \alpha - d\rangle - k_1k_2\langle T, \alpha - d\rangle + k_1^2\langle N, \alpha - d\rangle + k_1\langle B, T\rangle]$$

then $f'''(0) = 0$ implies $k_1^2(0)\langle N_o, \alpha(0) - d\rangle = 0$. And we obtain $\langle N_o, \alpha(0) - d\rangle = 0$

.

Now we investigate the numbers u_1, u_2, u_3 such that $\alpha(0) - d = u_1T_o + u_2N_o + u_3B_o$. Considering $\langle T_o, \alpha(0) - d\rangle = 0$, $\langle B_o, \alpha(0) - d\rangle = 0$, $\langle N_o, \alpha(0) - d\rangle = 0$ and Frenet frame, we have $\langle T_o, \alpha(0) - d\rangle = u_1\langle T_o, T_o\rangle + u_2\langle T_o, N_o\rangle + u_3\langle T_o, B_o\rangle$ then $u_2 = 0$. $\langle N_o, \alpha(0) - d\rangle = u_1\langle N_o, T_o\rangle + u_2\langle N_o, N_o\rangle + u_3\langle N_o, B_o\rangle$ then $u_1 = 0$. Similarly $\langle B_o, \alpha(0) - d\rangle = u_1\langle B_o, T_o\rangle + u_2\langle B_o, N_o\rangle + u_3\langle B_o, B_o\rangle$
then $u_3 = 0$. Thus $d = \alpha(0)$. Since $\langle \alpha(0) - d, \alpha(0) - d\rangle = r^2$ then we must have $r = 0$. Also the equation of the pseudosphere which contacts at the third order to α at $\alpha(0)$ is $\langle y - \alpha(0), y - \alpha(0)\rangle = 0$. □

We can state the following theorems for the osculating sphere of a null curve in $\mathbb{R}_1^4$ and $\mathbb{R}_2^4$ as follows:

Theorem 3.2. *Let $k_1(0) \neq 0$. The sphere that contacts at the third order to null (lightlike) curve $\alpha : I \to \mathbb{R}_1^4$ at $\alpha(0)$ is the pseudosphere at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ is*

$$-2x_1x_2 + x_3^2 + (x_4 - \frac{1}{k_3})^2 = \left(\frac{1}{k_3}\right)^2. \tag{3.2}$$

Theorem 3.3. *Let $k_1(0) \neq 0$. The sphere that contacts at the fourth order to null (lightlike) curve $\alpha : I \to \mathbb{R}_2^4$ at $\alpha(0)$ with timelike first binormal B_1 is the pseudosphere at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ is*

$$-2x_1x_2 - x_3^2 + (x_4 - \frac{1}{k_3})^2 = \left(\frac{1}{k_3}\right)^2. \tag{3.3}$$

Theorem 3.4. *Let $k_1(0) \neq 0$. The sphere that contacts at the fourth order to null (lightlike) curve $\alpha : I \to \mathbb{R}_2^4$ at $\alpha(0)$ with timelike second binormal B_2 is the pseudosphere at $\alpha(0)$ and the equation of the sphere according to the frame*

$\{T_o, N_o, B_{1_0}, B_{2_0}\}$ *is*

$$-2x_1x_2 + x_3^2 - (x_4 - \frac{1}{k_3})^2 = \left(\frac{1}{k_3}\right)^2. \tag{3.4}$$

4. Osculating Sphere of a Spacelike Curve

Theorem 4.1. *Let $k_1(0)$ and $k_2(0)$, the curvatures of a spacelike curve $\alpha : I \to \mathbb{R}_1^3$ with timelike principal normal N at $\alpha(0)$, be different from zero. Then there exists a sphere which touches at the third order to the curve at $\alpha(0)$, and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}\}$ is*

$$x_1^2 - (x_2 + \rho_o)^2 + (x_3 - \rho_o'\sigma_o)^2 = -\rho_o^2 + (\rho_o'\sigma_o)^2, \tag{4.1}$$

where $\rho_o = \dfrac{1}{k_1(0)}$ and $\sigma_o = \dfrac{1}{k_2(0)}$.

Proof. Let $f(t) = \langle \alpha(t) - d, \alpha(t) - d\rangle - r^2$. If $f(0) = 0$ then $\langle \alpha(0) - d, \alpha(0) - d\rangle = r^2$. Since $f' = 2\langle \alpha', \alpha - d\rangle$ then $f'(0) = 0$ implies to $\langle T_o, \alpha(0) - d\rangle = 0$. Since $f'' = 2[\langle \alpha'', \alpha - d\rangle + \langle \alpha', \alpha'\rangle], f''(0) = 0$ implies to $[\langle k_1(0)N_o, \alpha(0) - d\rangle + \langle T_o, T_o\rangle] = 0$ and we get $\langle N_o, \alpha(0) - d\rangle = -\rho_o$. Since $f''' = 2[\langle \alpha''', \alpha - d\rangle + 3\langle \alpha'', \alpha'\rangle]$ and the equality $f'''(0) = 0$ implies to

$$\langle k_1^2(0)T_o + k_1'(0)N_o + k_1(0)k_2(0)B_o, \alpha(0) - d\rangle + 3\langle k_1(0)N_o, T_o\rangle = 0$$

Let us consider $\langle T_o, \alpha(0) - d\rangle = 0$, $\langle N_o, \alpha(t) - d\rangle = -\rho_o$, $\langle N_o, T_o\rangle = 0$ then we obtain

$$\langle B_o, \alpha(0) - d\rangle = \frac{\rho_o k_1'(0)}{k_1(0)k_2(0)} = \frac{k_1'(0)}{k_1^2(0)k_2(0)} = -\rho_o'\sigma_o.$$

Now we investigating the numbers u_1, u_2, u_3 such that $\alpha(0) - d = u_1T_o + u_2N_o + u_3B_o$, we obtain

$$\alpha(0) - d = \rho_o N_o - \rho_o'\sigma_o B_o.$$

Thus, the origin of the sphere which contacts at the third order to the curve α at point $\alpha(0)$ is

$$d = \alpha(0) - \rho_o N_o + \rho_o'\sigma_o B_0.$$

When a variable P is given on this sphere, we suppose $P = \alpha(0) + x_1T_o + x_2N_o + x_3B_o$. Hence, we get

$$P - d = x_1T_o + (x_2 + \rho_o)N_o + (x_3 - \rho_o'\sigma_o)B_o,$$

then

$$\langle P - d,\ P - d\rangle = x_1^2 - (x_2 + \rho_o)^2 + (x_3 - \rho_o'\sigma_o)^2.$$

Also $r^2 = \langle \alpha(0) - d, \alpha(0) - d\rangle = -\rho_o^2 + (\rho_o'\sigma_o)^2$ then the equation (4.1) is obtained. □

Corollary 2. *If $-\rho_o^2 + (\rho_o'\sigma_o)^2 > 0$ at $\alpha(0)$ for the spacelike curve $\alpha : I \to \mathbb{R}_1^3$ whose principal normal vector is timelike, then osculating sphere is a one-sheet hyperboloid. If $-\rho_o^2 + (\rho_o'\sigma_o)^2 < 0$, then osculating sphere is a two-sheet hyperboloid.*

Now, we show that the circle which is the intersection of the osculating sphere at $\alpha(0)$ for a spacelike curve $\alpha : I \to \mathbb{R}^3_1$ whose principal normal vector is timelike the plane $Sp\{T_o, N_o\}$, contacts at the second order to the curve at $\alpha(0)$. The circle is called osculating circle of the spacelike curve at $\alpha(0)$.

Theorem 4.2. *A spacelike curve $\alpha : I \to \mathbb{R}^3_1$ whose principal normal vector is timelike has a circle $\gamma : \mathbb{R} \to \mathbb{R}^3_1$ which contacts at the second order to the curve at $\alpha(0)$ and*

$$\gamma(\theta) = \alpha(0) + (\rho_o \sinh\theta)T_o + \rho_o(-1+\cosh\theta)N_o. \tag{4.2}$$

Proof. The equation of the intersection of the plane $Sp\{T_o, N_o\}$ with the sphere which is given in (4.1) according to the frame $\{T_o, N_o, B_o\}$ is

$$-x_1^2 + (x_2+\rho_o)^2 = \rho_o^2.$$

Then we have $x_1 = \rho_o \sinh\theta$, $x_2 = \rho_o(-1+\cosh\theta)$. Thus, the intersection circle can be given as in the equality in (4.2). Clearly $\gamma(0) = \alpha(0)$. Since

$$\gamma'(\theta) = (\rho_o \cosh\theta)T_o + (\rho_o \sinh\theta)N_o$$

then we get,

$$\gamma'(0) = \rho_o T_0 = \rho_o \alpha'(0).$$

Since $\gamma''(\theta) = (\rho_o \sinh\theta)T_o + (\rho_o \cosh\theta)N_o$, then we obtain,

$$\gamma''(0) = \rho_o N_o = \rho_o^2 \alpha''(0).$$

The equalities $\gamma(0) = \alpha(0)$, $\gamma'(0) = \rho_o\alpha'(0)$ and $\gamma''(0) = \rho_o^2\alpha''(0)$ show that the curve γ contacts at the second order to the curve α at $\alpha(0)$. □

Corollary 3. *Osculating circle of a spacelike curve $\alpha : I \to \mathbb{R}^3_1$ whose principal normal vector is timelike at $\alpha(0)$ is also a spacelike curve.*

Proof. It is easy to see that for every $\theta \in \mathbb{R}$, $\langle\gamma'(\theta), \gamma'(\theta)\rangle = \rho_o^2(\cosh\theta)^2 - \rho_o^2(\sinh\theta)^2 = \rho_o^2 > 0$. □

Theorem 4.3. *Let $\alpha : I \to \mathbb{R}^3_1$ be a spacelike curve with timelike binormal vector field and the curvatures of the curve at point $\alpha(0)$; $k_1(0)$ and $k_2(0)$ different from zero. Thus there exist a sphere which contacts at the third order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_o\}$ is*

$$x_1^2 + (x_2-\rho_o)^2 - (x_3 - \rho_o'\sigma_o)^2 = \rho_o^2 - (\rho_o'\sigma_o)^2, \tag{4.3}$$

where $\rho_o = \dfrac{1}{k_1(0)}$ and $\sigma_o = \dfrac{1}{k_2(0)}$.

Proof. The proof is similar to the proof of the Theorem 4.1. □

Theorem 4.4. *For each spacelike curve $\alpha : I \to \mathbb{R}^3_1$, whose binormal vector field is timelike, there exist a circle $\gamma : R \to \mathbb{R}^3_1$ which contacts at the second order to the curve at $\alpha(0)$ and*

$$\gamma(\theta) = \alpha(0) + \rho_o \sinh(\theta+\pi))T_o + (\rho_o + \rho_o\cosh(\theta+\pi)N_o. \tag{4.4}$$

Proof. The proof is similar to the proof of the Theorem 4.3. □

Corollary 4. *Osculating circle of a spacelike curve $\alpha : I \to \mathbb{R}^3_1$ whose binormal vector field is timelike at point $\alpha(0)$ is also a spacelike curve.*

Proof. It can be made in a similar way to the proof of Theorem4.2. □

We can state the following theorems for the osculating sphere of a spacelike curve in $\mathbb{R}^4_1$ and $\mathbb{R}^4_2$ as follows:

Theorem 4.5. *Let $\alpha : I \to \mathbb{R}^4_1$ be a spacelike curve with timelike principal vector field N and the curvatures of the curve at $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts at the fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 - (x_2 + \lambda_1)^2 + (x_3 - \lambda_2)^2 + (x_4 + \lambda_3)^2 = -\lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{4.5}$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0' \sigma_0$, $\lambda_3 = \left(\left(-(\rho_0' \sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$ *and* $\rho_0 = \dfrac{1}{k_1(0)}$, $\sigma_0 = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 4.6. *Let $\alpha : I \to \mathbb{R}^4_1$ be a spacelike curve with timelike first binormal B_1 and the curvatures of the curve at $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts at the fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 + (x_2 - \lambda_1)^2 - (x_3 + \lambda_2)^2 + (x_4 + \lambda_3)^2 = \lambda_1^2 - \lambda_2^2 + \lambda_3^2, \tag{4.6}$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0' \sigma_0$, $\lambda_3 = \left(\left(-(\rho_0' \sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$ and $\rho_0 = \dfrac{1}{k_1(0)}$, $\sigma_0 = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 4.7. *Let $\alpha : I \to \mathbb{R}^4_2$ be a spacelike curve with timelike principal normal N and timelike first binormal B_1 and the curvatures of the curve at point $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 - (x_2 + \lambda_1)^2 - (x_3 + \lambda_2)^2 + (x_4 - \lambda_3)^2 = -\lambda_1^2 - \lambda_2^2 + \lambda_3^2, \tag{4.7}$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0' \sigma_0$, $\lambda_3 = -\left(\left((\rho_0' \sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$ and $\rho_0 = \dfrac{1}{k_1(0)}$, $\sigma_0 = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 4.8. *Let $\alpha : I \to \mathbb{R}^4_2$ be a spacelike curve with timelike principal normal N and timelike second binormal B_2 and the curvatures of the curve at point $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 - (x_2 + \lambda_1)^2 + (x_3 - \lambda_2)^2 - (x_4 + \lambda_3)^2 = -\lambda_1^2 + \lambda_2^2 - \lambda_3^2, \quad (4.8)$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0'\sigma_0$, $\lambda_3 = -\left(\left((\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$ and $\rho_0 = \frac{1}{k_1(0)}$, $\sigma_0 = \frac{1}{k_2(0)}$, $\omega_0 = \frac{1}{k_3(0)}$.

Theorem 4.9. *Let $\alpha : I \to \mathbb{R}_2^4$ be a spacelike curve with timelike first binormal B_1 and timelike second binormal B_2 and the curvatures of the curve at point $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts at the fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 + (x_2 - \lambda_1)^2 - (x_3 + \lambda_2)^2 - (x_4 - \lambda_3)^2 = \lambda_1^2 - \lambda_2^2 + \lambda_3^2, \quad (4.9)$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0'\sigma_0$, $\lambda_3 = \left(\left(-(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$ and $\rho_0 = \frac{1}{k_1(0)}$, $\sigma_0 = \frac{1}{k_2(0)}$, $\omega_0 = \frac{1}{k_3(0)}$.

ÖZET. Üç boyutlu Öklid uzayında bir $\alpha : I \to R^3$ eğrisinin $\alpha(0)$ noktasında eğriye üçüncü basamaktan değen bir ve yalnız bir küre vardır. Oskülatör düzlemiyle bu kürenin arakesiti, eğriye $\alpha(0)$ noktasında ikinci basamaktan değen bir çemberdir [5]. Bu çalışmada R_1^3, R_1^4 ve R_2^4 yarı Riemann uzaylarında zamansı, uzaysı ve boşluksu (ışıksı) eğrilerin her biri için eğrinin oskülatör küresi ve eğrilik çemberi incelenmiştir.

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