

CELLULAR FOLDING

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ABSTRACT

In this paper we introduce the notion of cellular and neat cellular foldings on a category of complexes equipped with cellular subdivision such that each closed n -cell is homeomorphic to a closed Euclidean n -cell. Then we obtained the necessary and sufficient conditions for a cellular map to be a cellular folding and a neat cellular folding respectively.

1. INTRODUCTION

Let K and L be directed complexes and $f: |K| \rightarrow |L|$ be continuous function. Then $f: K \rightarrow L$ is a cellular function if

- (1) for each directed cell $\sigma \in K$, $f(\sigma) = \pm \tau$ where τ is a directed cell in L ,
- (2) $\dim(f(\sigma)) \leq \dim(\sigma)$, [4].

Let K and L be complexes of the same dimension n and K be equipped with finite cellular subdivision such that each closed n -cell is homeomorphic to a closed Euclidean n -cell.

A cellular map $\Phi: K \rightarrow L$ is a **cellular folding** iff Φ satisfies the following:

- (i) For each i -cell $e^i \in K$, $\Phi(e^i)$ is an i -cell in L . ie. Φ maps i -cells to i -cells.
- (ii) If \bar{e} contains n vertices, then $\overline{\Phi(e)}$ must contains n distinct vertices.

In the case of directed complexes it is also required that Φ maps directed i -cells of K to i -cells of L but of the same orientation.

A cellular folding $\Phi: K \rightarrow L$ is a **neat cellular folding** if $L^n - L^{n-1}$ consists of a single n -cell. Int L .

The set of complexes together with the neat cellular foldings form a category which is a subcategory of the category of cellular foldings and we denote it by $N(K,L)$. Thus if $N(K,L) \neq \Phi$, then $\dim L \geq \dim K$.

Throughout this paper, we use the term complex to mean a complex equipped with cellular subdivision such that each closed n -cell is homeomorphic to a closed Euclidean n -cell.

2. CHAIN MAPS AND CELLULAR FOLDING

The next theorem gives the necessary and sufficient condition for a cellular map to be a cellular folding.

THEOREM 1. Let K and L be complexes of the same dimension n and $\Phi: K \rightarrow L$ be a cellular map such that $\Phi(K) = L \neq K$. Then Φ is a cellular folding iff the map $\Phi_p: C_p(K) \rightarrow C_p(L)$ between chain complexes $(C_p(K), \partial_p)$, $(C_p(L), \partial'_p)$ is a chain map.

PROOF. Let Φ be a cellular folding, then it is a cellular map and we can define a homomorphism $\Phi_p: C_p(K) \rightarrow C_p(L)$ by

$$\Phi_p(\sigma) = \begin{cases} \Phi(\sigma) & \text{if } \Phi(\sigma) \text{ is a } p\text{-cell in } L, \\ \Phi & \text{if } \dim(\Phi(\sigma)) < p, \end{cases} \quad [4]$$

and since a cellular folding maps p -cells to p -cells, $\Phi_p(\sigma_\lambda)$ is a p -cell in L for all λ .

Thus for a p -chain $C = a_1\sigma_1^p + a_2\sigma_2^p + \dots + a_m\sigma_m^p \in C_p(K)$ where a_i 's $\in \mathbb{Z}$ and σ_i 's are p -cells in K ,

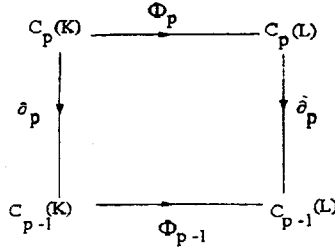
$$\Phi_p(C) = \Phi_p(a_1\sigma_1^p + a_2\sigma_2^p + \dots + a_m\sigma_m^p)$$

$$\Phi_p(C) = \Phi_p(a_1\sigma_1^p) + \Phi_p(a_2\sigma_2^p) + \dots + \Phi_p(a_m\sigma_m^p)$$

$$= a_1\Phi_p(\sigma_1^p) + a_2\Phi_p(\sigma_2^p) + \dots + a_m\Phi_p(\sigma_m^p) \in C_p(L).$$

Now since the closure of both σ_λ^p and $\Phi(\sigma_\lambda^p)$ has the same number of distinct vertices, then $\Phi_{p-1} \circ \partial_p = \partial'_p \circ \Phi_p$, where $\partial_p: C_p(K) \rightarrow C_{p-1}(K)$

and $\partial'_p: C_p(L) \rightarrow C_{p-1}(L)$ are the boundary operators, that is to say the following diagram commutes



and hence Φ is a chain map.

Conversly, suppose Φ is not a cellular folding, then there exists a j -cell σ in K such that $\Phi(\sigma)$ is a m -cell in L , where $m \neq j$. Since Φ_p is a homomorphism from the p -th chain of K to the p -th chain of L , then

$$\Phi_j \left(\sum_{i=1}^{n-1} \lambda_i \sigma_i^{(j)} + \lambda_n \sigma \right) = \sum_{i=1}^{n-1} \lambda_i \Phi(\sigma_i^{(j)}) + \lambda_n \Phi(\sigma)$$

but $\Phi(\sigma)$ is not a j -cell, then Φ_j cannot be a j -chain map and hence our assumption is false and we have the result.

2.1. Examples

1- Let K be a complex such that $|K|$ is the infinite strip $\{(x,y): -\infty < x < \infty, 0 \leq y \leq 2\}$ equipped with an infinite number of 2-cells such that the closure of each 2-cell consists of four 0-cells and four 1-cells, see Fig. 1.

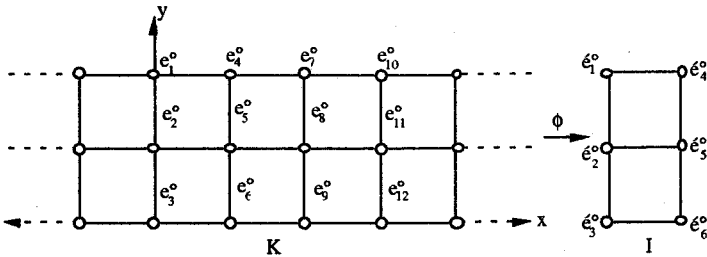


Fig. 1

Let L be a complex with six 0-cells, seven 1-cells and two 2-cells. The cellular map $\Phi: K \rightarrow L$ defined by

$$\Phi(e_n^0) = e_m^0 \begin{cases} \text{where } m = 1,2,\dots,6, & \text{and} \\ n - m \text{ is a multiple of } 6 \end{cases}$$

$$\Phi(e_i^2) = \begin{cases} e_1^2 & \text{if } i \text{ is odd} \\ e_2^2 & \text{if } i \text{ is even} \end{cases}$$

This map is a cellular folding.

2- Consider a complex K such that $|K| = S^2$ with cellular subdivision consisting of two 0-cells, four 1-cells and four 2-cells. Let $\Phi: K \rightarrow K$ be a cellular map defined by

$$\Phi(e_1^0, e_2^0) = (e_1^0, e_2^0)$$

$$\Phi(e_2^1, e_4^1) = (e_1^1, e_2^1)$$

$$\Phi(e_n^2, e_1^2) = e_1^2 \text{ for } n = 1,2,3,4.$$

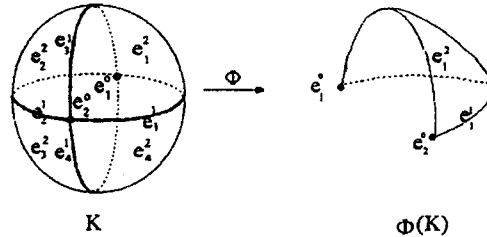


Fig. 2

This map is a cellular folding with image consisting of two 0-cells, two 1-cells and a single 2-cell, see Fig. 2.

3- Consider a complex K such that $|K|$ is a torus with cellular subdivision consisting of three 0-cells, six 1-cells and three 2-cells.

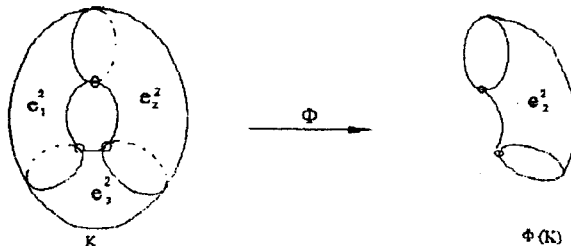


Fig. 3

Any cellular map $\Phi: K \rightarrow K$ which has two vertices in the image is not a cellular folding since Φ_1 is not a chain map in this case.

4- Consider a complex K such that $|K|$ is a torus with cellular subdivision consisting of four 0-cells, eight 1-cells and four 2-cells.

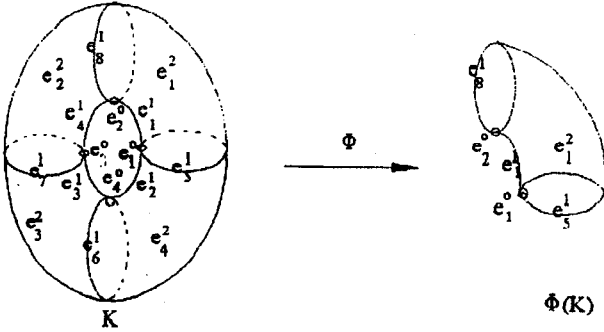


Fig. 4

A cellular map $\Phi: K \rightarrow K$ defined by

$$\begin{aligned} \Phi(e_1^0, e_2^0, e_3^0, e_4^0) &= (e_1^0, e_2^0, e_3^0, e_4^0) \\ \Phi(e_1^1, e_2^1, \dots, e_8^1) &= (e_1^1, e_1^1, e_1^1, e_1^1, e_3^1, e_5^1, e_5^1, e_8^1) \\ \Phi(e_n^2) &= e_1^2 \text{ for } n = 1, 2, 3, 4. \end{aligned}$$

This map is a cellular folding with image consisting of two 0-cells, three 1-cells and a single 2-cell.

5- Consider a cell-complex K such that $|K| = S^2$ with cellular subdivision consisting of four 0-cells, six 1-cells and four 2-cells, see Fig. 5.

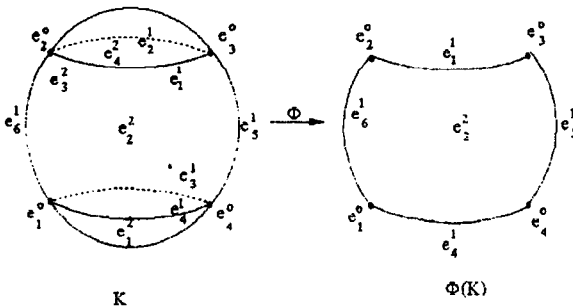


Fig. 5

Let $\Phi: K \rightarrow K$ be a cellular map defined by

$$\Phi(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_2^0, e_3^0, e_4^0)$$

$$\Phi(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1) = (e_1^1, e_1^1, e_4^1, e_4^1, e_5^1, e_6^1)$$

$$\Phi(e_n^2) = e_n^2, \quad n = 1, 2, 3, 4.$$

This map is not cellular folding since $\overline{e_1^2}, \overline{\Phi(e_1^2)}$ does not contain the same number of vertices.

3. NEAT CELLULAR FOLDING

The following theorem gives necessary and sufficient condition for a cellular map to be a neat cellular folding.

THEOREM (2). If $\Phi \in N(K, L)$ such that $\Phi(K) = L \neq K$, then Φ is a neat cellular folding iff the map $\Phi: C_p(K) \rightarrow C_p(L)$ between the chain complexes $(C_p(K), \partial_p)$, $(C_p(L), \partial'_p)$ is a chain map and $H_p(K) \simeq \ker \Phi_p^*$, where $\Phi_p^*: H_p(K) \rightarrow H_p(L)$, $p \geq 1$ is the induced homomorphism.

PROOF. Assuming that Φ is a neat cellular folding, then it is a cellular folding and hence the map $\Phi: H_p(K) \rightarrow H_p(L)$ between the chain complexes $(C_p(K), \partial_p)$, $(C_p(L), \partial'_p)$ is a chain map. Now consider the induced homomorphism $\Phi_p^*: H_p(K) \rightarrow H_p(L)$, there is a short exact sequence:

$$0 \rightarrow \ker \Phi_p^* \xrightarrow{i^*} H_p(K) \xrightarrow{\Phi_p^*} \text{Im } \Phi_p^*$$

where i^* is the induced homomorphism by the inclusion. Since Φ is surjective, we have $\text{Im } \Phi_p^* \simeq H_p(L)$, but $H_p(L) = 0$ for neat cellular folding, hence the above sequence will take the form:

$$0 \rightarrow \ker \Phi_p^* \xrightarrow{i^*} H_p(K) \rightarrow 0.$$

The exactness of this sequence implies that

$$H_p(K) \simeq \ker \Phi_p^*$$

Conversely, suppose Φ is a chain map between chain complexes and $H_p(K) \simeq \ker \Phi_p^*$ but Φ is not neat, then $L^n - L^{n-1}$ consists of more than one n -cells. Thus

$$H_0(L) \simeq Z^j, \quad H_p(L) = 0, \quad \text{for } p = 1, 2, \dots, n$$

and

$$H_p(K) \simeq H_p(L) \oplus \ker \Phi_p^* \neq \ker \Phi_p^* \quad \text{for } p = 0$$

hence the assumption is false and Φ is neat.

It should be noted that examples (2) and (4) are neat cellular foldings.

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