Commun. Fac. Sci. Univ. Ank. Series A1
V. 47. pp. 121-124 (1998)

UNIFORMISATION OF RIEMANN SURFACES AND TODA CHAIN

E. SOYTÜRK

Department of Mathematics, Faculty of Sciences, University of Ankara, Ankara, TURKEY

(Received March 20, 1998; Accepted July 31, 1998)

ABSTRACT

In this article using the Schottky uniformisation of the hyperelliptic Riemann surface new formulae for the characteristic of finite zone solutions to the equations of Toda chain are obtained. For this characteristic of the solution are expressed through the parameters of uniformisation with the use of Poincaré series.

1. INTRODUCTION

In [2] the finite zone solutions of Korteweg-de Vries (KdV) equation and Toda chains were determined by using the theory of hyperelliptic Riemann surfaces. In this article the characteristics of the solutions were given by using Abel differentials. However the calculations of characteristics by using Abel differentials caused some difficulties in application. In [1], for the characteristics of KdV equations, an effective formula by using the Schottky uniformisation of Riemann surfaces was found. In this article using the similar method given in [1], we obtain a new formula for the characteristics of finite zone solutions for Toda chains with respect to Poincaré series.

2. SCHOTTKY UNIFORMISATION AND TODA CHAIN

Let us consider the Riemann surface of hyperelliptic curve Γ given by

$$w^2 = \prod_{i=1}^{2N+2} (z - z_i) \ , \ z_1 < z_2 < \dots < z_{2N+2} < \infty \ .$$

We will denote the points of Γ of the form $P^{\pm} = (\infty, \pm)$ by P^+ and P^-, where $\pm$ refer to the upper and lower branches of the curve Γ, respectively. Let a_1, b_1, ..., a_N, b_N be some canonical basis of cycles of

Γ, du_n be the normed differentials of first kind (i.e. holomorphic differentials), Ω_{P^+}, Ω_{P^-} be the normed differentials of second kind with the dual pole at the points P^+, P^- respectively. It is clear that

$$du_n = f_n\left(z^{\pm}\right) dz^{\pm}\ , \int_{a_n} du_m = 2\pi i \delta_{nm}\ , (n,\ m = 1,\ ...,\ N)\ ,$$

where $z^{\pm}$ are the local parameters in the neighbourhood of the points $P^{\pm}$ and f_1, ..., f_N are holomorphic functions, [3].

The matrix $B = (B_{nm})_{n,m=1}^{N}$ is called the period matrix of Γ, where

$$B_{nm} := \int_{b_n} du_m\ .$$

Let the characteristics of Abel differentials be

$$U_n := \int_{P^-}^{P^+} du_n\ , \tag{1}$$

$$2V_n = \oint_{b_n} \Omega_{P^+} + \oint_{b_n} \Omega_{P^-}\ , \tag{2}$$

and let

$$\theta(z \backslash B) = \sum_{m \in Z^N} \exp\left\{\frac{1}{2}\langle B_m,\ m\rangle + \langle z,\ m\rangle\right\}$$

be the Riemann theta function. Then the finite zone solution of Toda chain

$$\begin{aligned} \dot{\upsilon}_n(t) &= C_{n+1}(t) - C_n(t) \\ \dot{C}_n(t) &= C_n(t)\left[\upsilon_n(t) - \upsilon_{n-1}(t)\right] \end{aligned} \tag{3}$$

is the following [2],

$$\upsilon_n(t) = \frac{d}{dt} \ln \frac{\theta\left[(n+1)U + tV + z_0\right]}{\theta\left[nU + tV + z_0\right]}\ , \tag{4}$$

$$C_n(t) = \frac{\theta\left[(n+1)U + tV + z_0\right]\ \theta\left[(n-1)U + tV + z_0\right]}{\theta^2\left[nU + tV + z_0\right]} \tag{5}$$

where z_0 is an arbitrary vector, the vectors $U = (U_1,\ ...,\ U_N)$, $V = (V_1,\ ...,\ V_N)$ are determined by the expressions (1), (2) and are known as the characteristics of the solutions (4), (5). Let G be a Schottky group generated by the collections of elements σ_1, σ_2, ..., σ_n. We show the fixed points of transformations σ_n with A_n and $(-A_n)$. Then we also show the subgroup of G generated by σ_n with G_n. Hence

$$\frac{\sigma_n z + A_n}{\sigma_n z - A_n} = \mu_n \frac{z + A_n}{z - A_n}, \quad 0 < \mu_n < 1 , \; A_n , \mu_n \in R .$$

It is obvious that $G \subset SL(2, R)$. Let G/G_n and $G_m \backslash G/G_n$ be factor group and the two sided factor groups, respectively. In the view of the results in [1];

$$dU_n = \sum_{\sigma\in G/G_n} \left[\frac{1}{z - \sigma(-A_n)} - \frac{1}{z - \sigma(A_n)}\right] dz \tag{6}$$

The Poincaré series is the holomorphic differentials of Γ. The components of the matrix B are defined as follows:

$$B_{nm} = \sum_{\sigma\in G_m \backslash G/G_n} \ln \left[\frac{A_m - \sigma(A_n)}{A_m - \sigma(-A_n)}\right]^2 \tag{7}$$

$$B_{nn} = \ln \mu_n + \sum_{\sigma\in G_n \backslash G/G_n,\ \sigma\neq 1} \ln \left[\frac{A_n - \sigma(A_n)}{A_n - \sigma(-A_n)}\right]^2 \tag{8}$$

Here

$$\sigma z = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in R, \qquad \alpha\delta - \beta\gamma = 1$$

Now, using (6) and (8), let us calculate the characteristics defined in (1) and (2).

$$U_n = \int_{P^-}^{P^+} du_n$$

$$= \int_{\beta/\delta}^{\alpha/\gamma} \sum_{\sigma\in G/G_n} \left[\frac{1}{\sigma z + A_n} - \frac{1}{\sigma z - A_n}\right] d\sigma z$$

$$= \sum_{\sigma\in G/G_n} \int_{\beta/\delta}^{\alpha/\gamma} \frac{d\sigma z}{\sigma z + A_n} - \sum_{\sigma\in G/G_n} \int_{\beta/\delta}^{\alpha/\gamma} \frac{d\sigma z}{\sigma z - A_n}$$

$$= \sum_{\sigma\in G/G_n} \left[\ln \frac{\frac{\alpha}{\gamma} + A_n}{\frac{\beta}{\delta} + A_n} - \ln \frac{\frac{\alpha}{\gamma} - A_n}{\frac{\beta}{\delta} - A_n}\right]$$

$$= \sum_{\sigma\in G/G_n} \ln \left[\left(\frac{-\delta A_n + \beta}{\gamma A_n - \alpha}\right) / \left(\frac{-\delta(-A_n) + \beta}{\gamma(-A_n) - \alpha}\right)\right]$$

$$= \sum_{\sigma\in G/G_n} \ln \frac{\sigma^{-1}(A_n)}{\sigma^{-1}(-A_n)}$$

$$= \sum_{\sigma\in G/G_n} \ln \frac{\sigma(A_n)}{\sigma(-A_n)},$$

then, we obtain

$$U_n = \sum_{\sigma\in G/G_n} \ln \frac{\sigma(A_n)}{\sigma(-A_n)}. \tag{9}$$

Since

$$\oint_{b_n} \Omega_{P^\pm} = f_n(z^\pm),$$

we have

$$2V_n = \oint_{b_n} \Omega_{P^+} + \oint_{b_n} \Omega_{P^-}$$

$$= f_n(0) + f_n(\infty)$$

$$= f_n(0)$$

$$= \sum_{\sigma\in G/G_n} \{\sigma(A_n) - \sigma(-A_n)\},$$

or

$$V_n = \frac{1}{2} \sum_{\sigma\in G/G_n} \{\sigma(A_n) - \sigma(-A_n)\}. \tag{10}$$

Thus the following theorem is proved:

Theorem. The finite zone solutions of (3) are (4) and (5). The characteristics of U and V are (9) and (10) and the elements of B are obtained by (7) and (8).

REFERENCES

[1] BOBENKO, A.I. and KUBENSKII, D.A., Qualitative analysis and calculations of finite group solutions of the Korteweg-de Vries equation Automorphic approach, Theoretical and Mathematical Physics, vol. 72 (1987) 929-935.

[2] DUBROVIN, B.A., MATHVEEV, V.B. and NOVIKOV, S.P., Non linear equations of Korteweg-de Vries type finite zone linear operators and Abelian Varieties, Russian Math. Surveys 3 (1976) 59-146.

[3] SPRINGER, G., Introduction to Riemann Surfaces, Addison-Wesley, New-York, (1957).