

ON A LAGUERRE INVERSION

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ABSTRACT

The existence of some automorphisms on certain Laguerre plane L is used to determine the algebraic structure of the projective plane, which is a completion of the affine plane L_A derived at a known point A . In this paper, firstly a special Laguerre automorphism, that is a Laguerre inversion, is defined; and then the set of fixed points and the set of fixed circles of the Laguerre inversion are investigated.

1. INTRODUCTION

A Laguerre plane is a system $(\mathcal{P}, \mathcal{C}, \in)$ which consists of a nonempty set \mathcal{P} of points, a nonempty set \mathcal{C} of subsets (circles) of \mathcal{P} and \in the set theoretical inclusion satisfying the following axioms [2], [4]:

L.1. For every three pairwise nonparallel points A, B, C , there is a unique circle containing them. This circle is denoted by (ABC) .

(Two points $A, B \in \mathcal{P}$ are said to be parallel if and only if $A=B$ or there is no such a circle ζ that $A, B \in \zeta$. This relation is an equivalence relation on the point set \mathcal{P} . Its equivalence classes are called generators. If A and B are parallel points, then we write $A//B$.)

L.2. For every point there exists a unique generator, containing this point.

L.3. Every circle intersects every generator, containing this point.

L.3. Every circle intersects every generator in exactly one point.

L.4. For each circle ζ , each point A on ζ , and each point $B \neq A$ in $L \setminus \zeta$, there is a unique circle ζ' containing A and B such that $\zeta \cap \zeta' = \{A\}$ ζ and ζ' are called tangent.

L.5. Every circle ζ contains at least three points and $L \setminus \zeta \neq \emptyset$.

The origin of the Laguerre geometry is the geometry of oriented lines and oriented circles with nonnegative radius of the Euclidean plane [2]. In [1], Benz constructed the following class of Laguerre planes. Let F be an arbitrary field and $V=F^3$ denote the three dimension vector space over F .

Let \mathcal{O} be an oval in the plane $\{(x,y,z) \in V: z=0\}$, i.e., is a subset of a projective plane such that i) each line cuts \mathcal{O} in at most two points, and ii) through each point $P \in \mathcal{O}$ there exists exactly one tangent, i.e. a line intersecting \mathcal{O} in exactly one point. Then $(\mathcal{P}, \mathcal{Q}, \epsilon)$ with $\mathcal{P} = \{(x,y,z) \in V: (x,y,0) \in \mathcal{O}\}$ and $\mathcal{Q} = \{(x,y,z) \in \mathcal{P}: z = ax+by+c, a,b,c \in F\}$ is a Laguerre plane in the above sense. Here \mathcal{P} is the set of points of an ovoidal cylinder and a circle is intersection of \mathcal{P} with a plane which is not parallel to the axis of the cylinder (For a general definition, see [4]). In [2], Benz shows that if \mathcal{O} is given by $x^2+y^2 = 1$ then the corresponding Laguerre plane is isomorphic to the parabolic model $(\mathcal{P}, \mathcal{Q}, \epsilon)$ over \mathcal{R} where $\mathcal{P} = \mathcal{R}^2 \cup \mathcal{R}$,

$$\mathcal{Q} = \{ \{(x,y) \in \mathcal{R}^2: y = ax^2+bx+c\} \cup \{a\}: a,b,c \in \mathcal{R} \}$$

and \mathcal{R} is the set of real numbers. In order to obtain some new Laguerre planes, Hartman replaced the parabolas in the above model by some particularly chosen curves [3].

The following theorem explains the main property of the Laguerre planes.

Theorem 1. Let \bar{X} be the generator on the point X . Then the geometrical structure,

$$L_A = (\mathcal{P}\bar{A}, \{\zeta \setminus \{A\}: A \in \zeta, \zeta \in \mathcal{Q}\} \cup \{\bar{X}: X \in \mathcal{P}, X \neq A\}, \epsilon)$$

is an affine plane.

This affine plane is called the affine plane derived by the point A of the Laguerre plane, [2].

2. HYPERBOLIC AND PARABOLIC PENCILS

The set of all circles containing nonparallel points A and B is called a hyperbolic pencil and denoted by

$$[A, B] = \{\zeta: A, B \in \zeta, \zeta \in \mathcal{C}\}$$

The set of all circles, which are tangent to the circle ζ at the given point A is called a parabolic pencil and denoted by $[A, \zeta]$,

$$[A, \zeta] = \{\zeta': \zeta' \in \mathcal{C}, \zeta' \cap \zeta = \{A\}\} \cup \{\zeta\},$$

[5].

Definition 1. Let $L = (\mathcal{P}, \mathcal{C}, \epsilon)$ and $L' = (\mathcal{P}', \mathcal{C}', \epsilon')$ be two Laguerre planes. If there exists a one to one and onto function f , which maps points and circles of L to points and circles of L' ; respectively, and if, $A \in \zeta \Rightarrow f(A) \in f(\zeta)$ for $\forall A \in \mathcal{P}$ and $\forall \zeta \in \mathcal{C}$, then these Laguerre planes are said to be isomorphic and such a function f is said to be isomorphism.

If $L = L'$ then the function f is called an automorphism of the plane L , [5]. The automorphism f maps parallel points to parallel points. Therefore an automorphism preserves the generators invariant in whole. Every automorphism f maps a hyperbolic pencil to a hyperbolic pencil and a parabolic pencil to a parabolic pencil, that is,

$$f([A, B]) = [f(A), f(B)]$$

$$f([A, \zeta]) = [f(A), f(\zeta)].$$

Restriction of an automorphism f to the affine plane is an automorphism η_{L_A} of L_A . Consider the affine plane L_A as an embedding into Laguerre plane L . If the restriction η_{L_A} acts on the points of the affine plane, then all points, which are parallel to A , are invariant under the mapping f .

Definition 2. An automorphism $f \neq I$ of a Laguerre plane is said to be Laguerre Inversion, if for all points $A \in \mathcal{P}$ with $A \neq f(A)$ it follows that A and $f(A)$ are not parallel and all the circles passing through A and $f(A)$ are invariant under f , where I denotes the identity automorphism.

Proposition 1. Every Laguerre inversion is an involution.

Proof. Let $A \neq f(A)$, then $A \# f(A)$. Since the Laguerre inversion f maps every circle of the hyperbolic pencil $[A, f(A)]$ into itself, the points $A, f(A)$ of all circles of the hyperbolic pencil $[A, f(A)]$ are invariant with respect to f . Then $A = f^2(A)$, that is f is an involution.

Proposition 2. Let A and B are two different points of a Laguerre plane L and f is a Laguerre inversion. Then the set $\{A, f(A), B, f(B)\}$ is either concircular or a subset of union of two generators.

Proof. Let A denote a point which is not fix under f . If some points B are not parallel to A and $f(A)$, then there is a circle ζ containing the points $A, f(A)$ and B . Since $\zeta \in [A, f(A)]$, $f(\zeta) = \zeta$ as according to the Definition 2. That is $f(B) \in \zeta$ and $\{A, f(A), B, f(B)\}$ consists of the points of one circle. Now, let $B // A$. Then $f(B) // f(A)$. If $B // f(A)$, then $f(B) // f^2(A) = A$, and by the Proposition 1, the set $\{A, f(A), B, f(B)\}$ is the union of two different generators. Finally, if A and B are two invariant points of f , then obviously the set $\{A, B\}$ is a subset of the union of two generators.

Proposition 3. If a generator \bar{A} is invariant under the Laguerre inversion f , then \bar{A} is pointwise invariant under f .

Proof. Let $\bar{A} = f(\bar{A})$. Then every point A of the generator \bar{A} can be transformed to the point $f(A)$, which is parallel to A . According to the Definition 2, f does not change any points of the generator \bar{A} . In the case of $f(A) = A$ we have $f(A) = \bar{A} = f(\bar{A})$. That is the generator \bar{A} is pointwise invariant under f .

Proposition 4. The set of fixed points of a Laguerre inversion is empty or consists of the points of one or two generators.

Proof. Assume that a Laguerre inversion f has the fixed point A . Then, according to Proposition 3, \bar{A} is pointwise invariant under f . If $B \notin \bar{A}$ is another fixed point of f , then the generators \bar{A} and \bar{B} ($\bar{B} \neq \bar{A}$) are pointwise invariant under f . If $R \notin \bar{A} \cup \bar{B}$ and $f(R) = R$, then the automorphism Π_{L_R} preserves pointwise invariant two different lines of the affine plane L_R . Then $\Pi_{L_R} = I_{L_R}$ and $f = I$. This is a contradiction to the condition of $f \neq I$.

Proposition 5. Let f be a Laguerre inversion. The set of all circles, that are invariant under a Laguerre inversion f and include a fixed point A , is a hyperbolic pencil when $f(A) \neq A$ and a parabolic pencil when $f(A) = A$.

Proof. Let $f(A) \neq A$. According to the Definition 2, the points A and $f(A)$ are supports of the hyperbolic pencil $[A, f(A)]$, which are preserved under f . Inversely, if the circle ζ is invariant under f and contains the point A , then for $A \in \zeta$ we have $f(A) \in f(\zeta) = \zeta$, then ζ is an element of hyperbolic pencil $[A, f(A)]$ and a fixed circle of f .

Now, let $f(A) = A$. In this case there will be two different situations.

a) The generator \bar{A} is unique generator which is pointwise invariant under f . If $B \notin \bar{A}$, then $B \neq f(B)$, that is, if $B \neq f(B)$, then A , B and $f(B)$ are not pairwise parallel points. Denote by ζ the circle which contains the points A , B and $f(B)$ and define the following set

$$K := \{\zeta \in \mathcal{C} : \zeta = (ABf(B)), \forall B \notin \bar{A}\}.$$

K is the set of circles which are invariant under f and covers the set $\mathcal{P}\bar{A}$. All circles of the set K which $f(B) \neq B$ necessarily contains a point $(B, f(B))$. Suppose that there are two different circles v and w from K , which have another common point $D \neq A$, besides A . Since v and w are invariant under f and A is a fixed point of f , then D is also a fixed point of f . But this contradicts to $D \neq A$. Therefore K is a parabolic pencil of fix circles under f with basepoint A .

b) f has two nonparallel fixed points A and D . Let $B \notin \bar{A} \cup \bar{D}$. Then $B \neq f(B)$ and the circle $\zeta = (ABf(B))$ is invariant under f . Let $S := [A, \zeta]$. Because of $f(A) = A$ and $\zeta = f(\zeta)$ the set S is invariant in whole under f . Since each circle $w \in S$ contains the fixed point $\bar{D} \cap w$, then the set S is elementwise invariant. In this case, A carries a parabolic pencil of fixed circles of f . Now let w be a circle, which contains A and invariant under f . Suppose that $[A, \zeta]$ is a parabolic pencil of fixed circles with carrier point A . We have proved above the existence of such a pencil.

Suppose that $w \notin [A, \zeta]$ and that $A \in w$, then $|w \cap v| = 2$ for every $v \in [A, \zeta]$ and $f(w \cap v) = w \cap v$. Therefore, both of elements of $w \cap v$ are also fixed points under f . Since this one is valid for every $v \in [A, \zeta]$, then w is pointwise invariant under f . According to the Proposition 3 and 4 a Laguerre inversion can not possess three pairwise nonparallel fixed points. Therefore $w \in [A, \zeta]$ and the Proposition 5 is proved.

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