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ON THE MEUSNIER'S THEOREM FOR LORENTZIAN SURFACES

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ABSTRACT

In the present paper we give an analog of the Meusnier's Theorem for Lorentzian surfaces in the Lorentzian space of the dimension 3.

1. INTRODUCTION

By L^3 we denote the space $\mathbf{R}^3$ endowed with the inner product $<,>$ of index 1 and call it Lorentzian 3–space. In L^3 every tangent space of a surface can be considered as a subspace of L^3 in a canonical way. Thus if a surface in L^3 has the tangent spaces of index 1 then we call the surface Lorentzian as in [4]. In addition, a curve in a Lorentzian surface called time–like, space–like or null whether its velocity vector is, [1].

In the Riemannian case, it is well known that all the curves pass through a point, say p, and have common and non asymptotic tangents at the point p have their curvature centers on a unique sphere and also have their curvature circles on another unique sphere. This fact known as the Meusnier's Theorem (see [2]). The essential part of this work devoted to give an analog of this fact in L^3.

Let α: $I \longrightarrow L^3$ be a unit speed curve in L^3 and $X = \dot{\alpha}$, where the notation dot indicates the derivative. If α is a space–like curve then there exist unique orthonormal vectors X, Y, Z, and the first and the second curvature functions k_1, k_2 from I to $\mathbf{R}$ such that

$$<X, X> = 1, \; <Y, Y> = -1, \; <Z, Z> = 1,$$

$$<X, Y> = <Y, Z> = <X, Z> = 0,$$

$$\left.\begin{aligned} D_X X &= k_1 Y \\ D_X Y &= k_1 X + k_2 Z \\ D_X Z &= k_2 Y \end{aligned}\right\} \qquad (1.1)$$

or

$$<X, X> = 1, \ <Y, Y> = 1, \ <Z, Z> = -1,$$
$$<X, Y> = <Y, Z> = <X, Z> = 0,$$
$$\left.\begin{aligned} D_X X &= k_1 Y \\ D_X Y &= -k_1 X + k_2 Z \\ D_X Z &= k_2 Y \end{aligned}\right\} \quad (1.2)$$

where Y is time–like or space–like. If the curve α is time–like then the unique orthonormal frame field $\{X, Y, Z\}$, exists such that

$$<X, X> = -1, \ <Y, Y> = <Z, Z> = 1,$$
$$<X, Y> = <Y, Z> = <Z, X> = 0,$$
$$\left.\begin{aligned} D_X X &= k_1 Y \\ D_X Y &= k_1 X + k_2 Z \\ D_X Z &= -k_2 Y \end{aligned}\right\} \quad (1.3)$$

where $\{X, Y, Z\}$ called Frenet frame field of α, [3].

We give the notion of curvature center as the following which is just as in the Euclidean case.

Definition 1. Let $\alpha: I \longrightarrow L^3$ be a non–null curve and $\{X, Y, Z\}$, k_1 are the Frenet frame field on α and the first curvature function of α. The point

$$C(t) = \alpha(t) + \frac{1}{k_1(t)} Y$$

is called the curvature center of α at the point $\alpha(t)$ and the pseudo 1– sphere centered at the point C (t) that lay on the plane spanned by X and Y called *curvature circle* of α at the point p.

Now, we recall a definition about plane sections, just as in the case of E^3, [2], as follows:

Definition 2. Let M be a Lorentzian surface in L^3 and Π a plane which passes through a point $p \in M$. If a tangent vector $X_p \in T_M(p)$ is in Π then the intersection curve $M \cap \Pi$ is called the section curve determined by X_p and if the plane Π is orthogonal to $T_M(p)$ then the section curve determined by X_p is called the *normal section curve* determined by X_p.

Finally,

Definition 3. Let $M \in L^3$ be a Lorentzian surface and X_p is a tangent vector to M at the point p. Let us denote a plane through X_p by π and the curvature center of the intersection curve of π and M, that is $M \cap \pi$, by C_i. The curve obtained by translating the curvature circle of the intersection curve $M \cap \pi$, at the point p, by the vector $\overrightarrow{C_i P}$ called conjugate curvature circle of the intersection curve $M \cap \pi$ at the point P.

2. THE MEUSNIER'S THEOREM FOR LORENTZIAN SURFACES

The main theorems are:

Theorem 1. Let M be a Lorentzian surface in L^3 and $p \in M$, $X_p \in T_M(p)$. We assume that $X_p \in T_M(p)$ is not an asymptotic direction on M then

i) The locus of the curvature centers of all the non-null section curves determined by X_p with space-like second Frenet vectors is a pseudosphere

ii) The locus of the fourth vertex point of the parallelogram which constructed with one diagonal $[CC_i]$ and three vertices P, C, C_i is a pseudo-sphere where C_i and C are the curvature centers of any section curve and the normal section curve determined by X_p, respectively.

Theorem 2. Let M be a Lorentzian surface in L^3 and $p \in M$, $X_p \in T_M(p)$. We assume that $X_p \in T_M(p)$ is not an asymptotic direction on M. Let the points C and C_i denote the curvature centers of the normal section curve and a section curve determined by X_p. Then,

i) All curvature circles of all the non-null section curves determined by X_p with space-like second Frenet vectors lie on a pseudo-sphere centered at the point C.

ii) All the conjugate curvature circles of all non-null section curves determined by X_p with time-like second Frenet vectors lie on a pseudo-sphere or a pseudo-hyperbolic space and the center of the pseudo-sphere or the hyperbolic space is the fourth vertex point of the parallelogram which is determined by the vertex points, p, C and C_i and one diagonal the line segment $[CC_i]$.

First of all we shall give the following Lemma.

Lemma 1. Let h be the second fundamental form of the Lorentzian surface M in L^3. If X_p is a tangent vector to M and V and k_1 are the second Frenet vector and the first curvature function of the section curve determined by X_p, respectively. Then

$$k_2(0) <V_p, N_p> = -h(X_p, X_p) \tag{2.1}$$

where N_p is the unit normal to M at the point p.

Proof is the same as in the E^3, so we don't give it here, (see, [5]).

If we consider the curve mentioned in the Lemma. 1. as the normal section curve determined by X_p then the equation (2.1) becomes

$$k_N(0) <V_p^N, N_p> = -h(X_p, X_p)$$

where we denote the curvature of that normal section curve α_N by $k_N(0)$ thus we get

$$k_N(0) = \begin{cases} h(X_p, X_p); \; V_p^N = -N_p; \text{ (that is, } \alpha_N \text{ is bending away from } N_p) \\ -h(X_p, X_p); \; V_p^N = N_p; \text{ (that is, } \alpha \text{ is bending forward } N_p) \end{cases} \tag{2.2}$$

where V_p^N denotes the second Frenet vector of α.

Now we use the term curvature radius which is the reciprocal of the curvature. So we conclude the following corollary.

Corollary: Let $\alpha: I \longrightarrow M$ be a curve on the Lorentzian manifold M and X_p is a non–asymptotic tangent vector to M. If g, g are the curvature radii of the normal section curve and a section curve determined by X_p, respectively, then

$$<V_2, N> = \frac{g}{g_N} = \frac{k_N}{k_1} \text{ when } <V_2^N, N> > 0$$

$$<V_2, N> = \frac{-g}{g_N} = \frac{-k_N}{k_1} \text{ when } <V_2^N, N> > 0$$

where V is the second Frenet vector of α and N is the unit normal vector field to M and k_1, k_N denote the curvatures of α and the normal section curve determined by X_p.

Finally we need the following two Lemmas for the proof of the Theorem 1 and the Theorem 2.

Lemma 2. Let $A, B \in L^3$ and the vector $\overrightarrow{AB}$ is space–like. Then the points p on the condition that

$$<\overrightarrow{PA}, \overrightarrow{PB}> = 0$$

are lies on a sphere $S_1^2(r)$, where the radius r is a constant and depends on the points A and B.

Proof: We choose an orthonormal basis $\{e_0, e_1, e_2\}$ for L^3 such that e_0 is a unit time–like vector. Thus, for any point $p \in L^3$ we have the following coordinate expression

$$\overrightarrow{OP} = x_0e_0 + x_1e_1 + x_2e_2$$

and we can identify the point p and the vector $\overrightarrow{OP}$ as well as

$$x_0e_0 + x_1e_1 + x_2e_2$$

and (x_0, x_1, x_2). Now, take

$$A = (a_0, a_1, a_2)$$
$$B = (b_0, b_1, b_2)$$
$$P = (x_0, x_1, x_2)$$

so

$$<\overrightarrow{AB}, \overrightarrow{AB}> = -(b_0 - a_0)^2 + (b_1 - a_1)^2 + (b_2 - a_2)^2 > 0. \quad (2.3)$$

If the point p satisfies the condition of the Lemma then; a direct computation shows that;

$$(x_0 - (1/2)(a_0 + b_0))^2 + (x_1 - (1/2)(a_1 + b_1))^2 + (x_2 - (1/2)(a_2 + b_2))^2 = c$$

where

$$c = (1/4)(-(b_0 - a_0)^2) + (b_1 - a_1)^2 + (b_2 - a_2)^2) + (1/2)(a_0 + b_0)^2$$

and because of (2.3) the constant c is positive. Thus what we get is that the point p lies on a sphere $S_1^2(\sqrt{c})$.

Lemma 3: Let M be a Lorentzian surface in L^3. If $p \in M$, $X_p \in T_M(p)$ and α is a section curve determined by X_p such that the second Frenet vector V_2 of α is time–like then the vector $\overrightarrow{PQ}$ is orthogonal to the vector $\overrightarrow{PC_i}$, where C_i is the curvature center of α at the point p and Q is the fourth vertex point of the parallelogram determined by the vertices p, C_i and C such that $[PQ]$ and $[CC_i]$ are diagonal s of the parallelogram and the point C is the curvature center of the normal section curve determined by X_p at the point p. Furthermore PQ is a space like vector (Figure. 1).

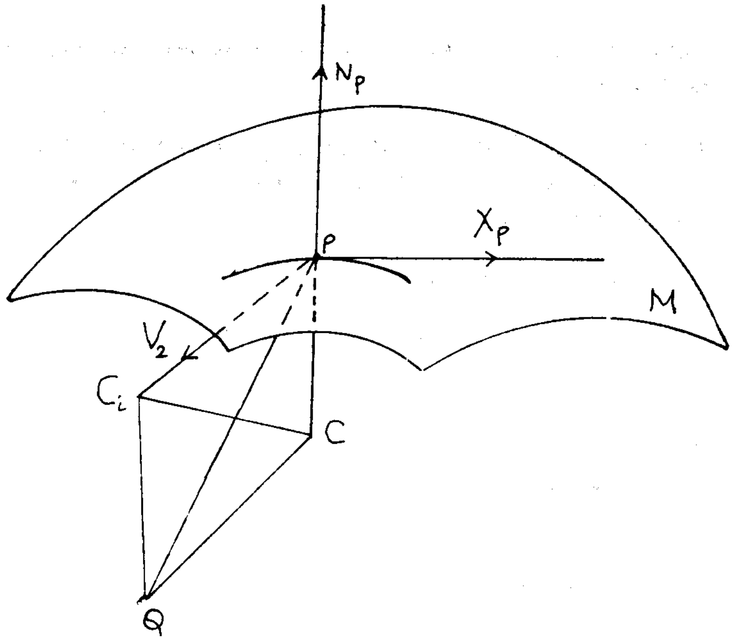

Figure. 1

Proof:

Let k_1 and k_N denote the first curvature of the section curve α and the normal section curve determined by X_p, respectively. So, in the case of $< V_2^N, N > \; > 0$, we have the following

$$C_i = p + \frac{1}{k_1} V_2$$

$$C = p + \frac{1}{k_N} N_p$$

where N_p is the unit normal to M at the point p (Figure. 1) (It should be noticed that if $< V_2^N, N > \; < 0$ then we have to take $N_p = - V_2^N$ that is,

$$C = P - \frac{1}{k_N} N_p$$

thus

$$\vec{PQ} = \frac{1}{k_1} V_2 + \frac{1}{k_N} N_p$$

and

$$<\vec{PQ}, \vec{PC_i}> = \frac{1}{k_1{}^2} <V_2, V_2> + \frac{1}{k_1} \frac{1}{k_N} <N_p, V_2>$$

since V_2 is a time–like curve and

$$<N_p, V_2> = \frac{k_N}{k_1}$$

by the corollary of Lemma. 1 so what we get is that

$$<\vec{PQ}, \vec{PC_i}> = 0$$

or

$$\vec{PQ} \perp \vec{PC_i}.$$

For the second assertion of the Lemma, since $\vec{PC_i}$ is a time–like vector and we proved that $\vec{PV} \perp \vec{PC_i}$ as above, so $\vec{PQ}$ is a space-like vector that completes the proof.

Proof of the Theorem 1. We will take the figure. 2 into account and assume that $<V_2{}^N, N_p> > 0$, thus

$$\vec{PC} = \frac{1}{k_N} N_p .$$

In the case of $<V_2{}^N, N_p> < 0$, we have to take the vector $\vec{PC}$ as $-(1 / k_N) N_p$. We would not deal with this possibility because, it makes no difference between the proofs that involving the signature of the number $<V_2{}^N, N_p>$. So we proceed the proof as follows

i) If V_2 is space–like then by the corollary we obtain

$$<gV_2 - g_N N_p, gV_2> = g^2 - gg_N (g / g_N) = 0.$$

On the other hand

$$\vec{PC_i} = gV_2$$

$$\vec{CC_i} = gV_2 - g_N N_p$$

so

$$<\vec{PC_i}, \vec{CC_i}> = 0$$

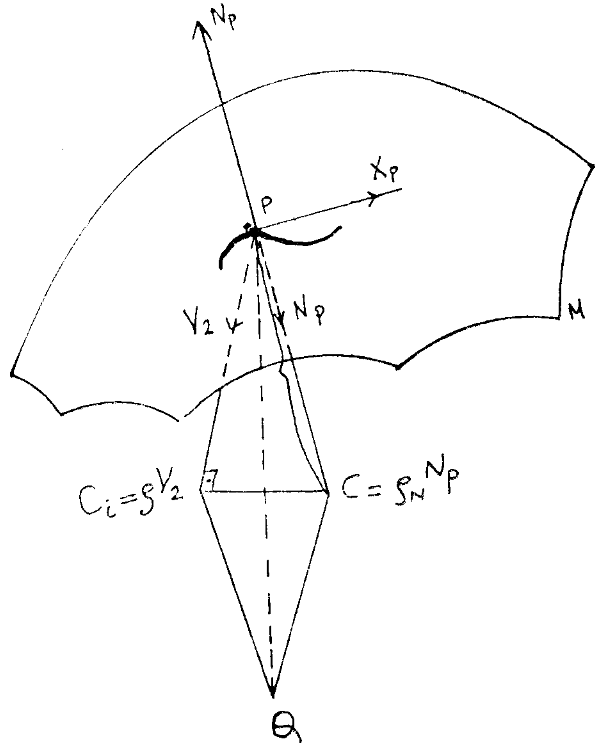

Figure. 2

that completes the proof of the assertion i) because of the Lemma. 2 (see. Fig. 1).

ii) If the second Frenet vector V_2 is time–like then;

$$\vec{PQ} = \vec{PC} + \vec{PC_i} = gV_2 + g_N N_p$$

$$\vec{CQ} = \vec{CP} + \vec{PQ} = gV_2$$

and by the corollary we obtain

$$< gV_2 + g_N N_p, gV_2 > = - g(g - g_N (g / g_N))$$

so

$$= 0$$

$$< \vec{PQ}, \vec{OC} > = 0$$

which completes the proof for the assertion ii) because of the Lemma 2.

Proof of the Theorem 2: Since C_i and C are curvature centers, we can write

$$C_i = p + \frac{1}{k_1} V_2$$

and

$$C = p + \frac{1}{k_N} N_p$$

where, k_1 and k_N are first curvature function of the section and the normal section curve determined by X_p. V_2 denotes the second Frenet vector of the section curve and N_p is the unit normal to M at the point p.

On the other hand, X_p is orthogonal to both $\overrightarrow{PC}$ and $\overrightarrow{PC_i}$ so the vector $\overrightarrow{CC_i}$ orthogonal to the vectors X_p and $\overrightarrow{PC_i}$ (figure. 3). Thus $\overrightarrow{CC_i}$ orthogonal to the plane spanned by the vectors $\overrightarrow{PC_i}$ and X_p at the point p.

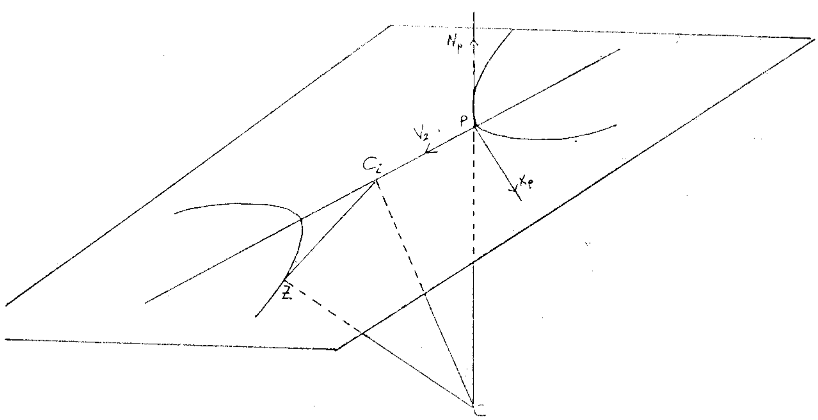

Figure. 3

(i) Let Z be a point that lies on the curvature circle at the point p of the section curve determined by X_p, Since $\overrightarrow{CC_i}$ is orthogonal to the plane spanned by $\overrightarrow{PC_i}$ and X_p and

$$\vec{ZC_i} \in S_p \ \{X_p, \vec{PC_i}\}$$

thus

$$< \vec{ZC}, \vec{ZC} > = < \vec{PC_i}, \vec{PC_i} > + < \vec{C_iC}, \vec{C_iC} > . \tag{2.4}$$

On the other and;

$$\vec{PC} = \vec{PC_i} + \vec{C_iC}$$

and so

$$< \vec{PC}, \vec{PC} > = < \vec{PC_i}, \vec{PC_i} > + < \vec{C_iC}, \vec{C_iC} > + 2 < \vec{PC_i}, \vec{C_iC} >$$

since; $\vec{C_iC} \perp \vec{PC_i}$ thus the right hand side of the above equation is the same as the right hand side of the equation (2.4) so

$$< \vec{PC}, \vec{PC} > = < \vec{ZC}, \vec{ZC} >$$

which means that, the point Z lies on the pseudo–sphere centered at the point C. Since Z is arbitrary that completes the proof of the assertion (i).

(ii) We will take the figure. 4 into account so we proceed the proof as follows

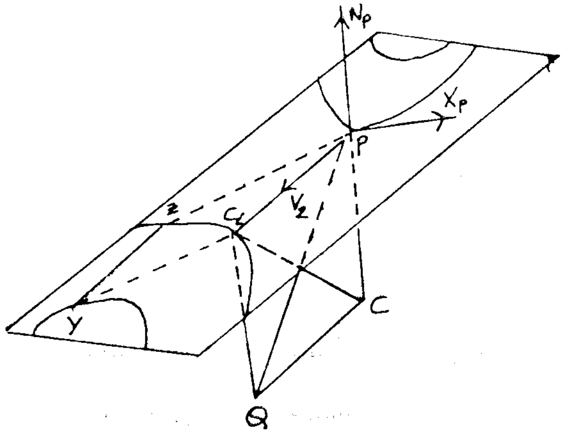

Figure. 4

Let Z be a point that lies on the special translated curvature circle of the section curve at the point p determined by X_p.

By Lemma. 3; $\overrightarrow{PQ}$ is orthogonal to $\overrightarrow{PC_i}$. Since $\overrightarrow{PQ}$ is a vector in the plane spanned by N_p and V_2 then $\overrightarrow{PQ}$ is orthogonal to the vectors V_2 and X_p so we obtain

$$< \overrightarrow{PQ}, \overrightarrow{PZ} > = 0 \tag{2.5}$$

so we get

$$< \overrightarrow{QZ}, \overrightarrow{QZ} > = < \overrightarrow{QP}, \overrightarrow{QP} > + < \overrightarrow{PZ}, \overrightarrow{PZ} > . \tag{2.6}$$

By the Definition. 3, there exists a point Y on the curvature circle at the point p determined by X_p, such that

$$\overrightarrow{YZ} = \overrightarrow{C_iP}$$

thus

$$\overrightarrow{C_iY} = \overrightarrow{PZ}. \tag{2.7}$$

Taking (2.7) into (2.6) we get

$$< \overrightarrow{QZ}, \overrightarrow{QZ} > = < \overrightarrow{QP}, \overrightarrow{QP} > + < \overrightarrow{C_iY}, \overrightarrow{C_iY} > \tag{2.8}$$

and since Y is a point on the curvature circle centered at C_i then

$$< \overrightarrow{C_iY}, \overrightarrow{C_iY} > = < \overrightarrow{PC_i}, \overrightarrow{PC_i} >$$

so by (2.8) we obtain

$$< \overrightarrow{QZ}, \overrightarrow{QZ} > = < \overrightarrow{QP}, \overrightarrow{QP} > + < \overrightarrow{PC_i}, \overrightarrow{PC_i} > \tag{2.9}$$

we recall that $\overrightarrow{QP}$ is a space-like, $\overrightarrow{PC_i}$ is a time-like so (2.9) can be written as the following form

$$< \overrightarrow{QZ}, \overrightarrow{QZ} > = || \overrightarrow{QP} ||^2 - || \overrightarrow{PC_i} ||^2$$

which completes the proof of the assertion (ii) since the pointz Z are lies on a pseudo-sphere or on a pseudo-hyperbolic space according to the sign of the number

$$\| \overrightarrow{QP} \|^2 - \| \overrightarrow{PC_i} \|^2.$$

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