

## CROSS-RATIOS OVER THE GEOMETRIC STRUCTURES WHICH ARE COORDINATIZED WITH ALTERNATIVE OR LOCAL ALTERNATIVE RINGS

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### ABSTRACT

In this note we summarize [3], [4] and [9] which are interested with cross ratio over alternative or local alternative rings. In [9], there is an algebraic expression which is not true [6], [7]. For this, we give some proofs by using formal calculations. And we give some basic results which are useful for calculations.

### 1. INTRODUCTION

When  $A$  is an alternative division ring with  $\text{char} \neq 2$ , the definition of cross-ratio over the projective line which is coordinatized by  $A$  is given by Schleiermacher (1965, [18]). In 1980, Ferrar [9] used this definition and did a similar work. Blunck (1991, [3]) "extended" this definition to the case of  $A$  having an arbitrary characteristic. In [6] and [7] definition of the cross-ratio is extended to the whole Mufang plane which is coordinatized by  $A$ .

More information about the geometric and algebraic preliminaries may be found in [10], [13], [15], [19] and [12], [16] respectively.

### 2. BASIC CONCEPTS

Let  $(\mathcal{P}, \mathcal{L}, \epsilon)$  be an incidence structure whose points and lines are denoted by  $P, Q, R, \dots$  and  $l, m, n, \dots$  respectively.

**Definition.** An incidence structure  $(\mathcal{P}, \mathcal{L}, \epsilon)$  is called a projective plane if the following axioms are satisfied:

- P1) Any two distinct points are incident with a unique line.
- P2) Any two distinct lines are incident with at least one point.
- P3) There exist a set of four points, no three collinear.

**Definition.** Let  $(\mathcal{P}, \mathcal{L}, \in)$  be incidence structure and “o” be equivalence relation on  $\mathcal{P}$  and on  $\mathcal{L}$ . We called “o” as neighbouring. Then  $M = (\mathcal{P}, \mathcal{L}, \in, O)$  is called projective Klingenberg plane (PK-plane), if the following axioms are satisfied:

PK1) Any two non-neighbouring points are incident with a unique line.

PK2) Any two non-neighbouring lines are incident with a unique point.

PK3) There is a projective plane  $M^* = (\mathcal{P}^*, \mathcal{L}^*, \in)$  and an incidence structure epimorphism  $\Psi: M \rightarrow M^*$  such that the conditions  $\Psi(P) = \Psi(Q) \Leftrightarrow P \square Q$ ,  $\Psi(l) = \Psi(m) \Leftrightarrow l \square m$  hold for all  $P, Q \in \mathcal{P}$ ,  $l, m \in \mathcal{L}$ .

In PK3,  $M^*$  is the canonic image of  $M$ .

If there exist a  $Q \in \mathcal{P}$  such that  $P \square Q$  and  $Q \in l$  then we say that a point  $P$  is near a line  $l$  and we denote this by  $P \square l$ .

In PK-planes, neighbouring points may have no, unique, or several joining lines. And neighbouring lines may have no, unique, or several intersection points.

**Definition.** Let  $M$  be a PK-plane.

(i) If any two points of  $M$  have at least one joining line then  $M$  is called punctally cohesive.

(ii) If any two lines of  $M$  have at least one intersection point then  $M$  is called lineary cohesive.

If  $M$  PK-plane is punctally and lineary cohesive than  $M$  is called cohesive.

A projective Hjelmslev plane is a cohesive PK-plane, such that any two points with exactly one joining line are nonneighbouring.

Now, we recall some notions for  $M$  PK-planes (cf. [2]):

If  $\sigma$  is an incidence structure automorphism such that preserving neighbour relation then  $\sigma$  is called a collineation of  $M$ . A centre and an axis of a collineation are defined as usual (cf. [17]). A  $(C, \alpha)$ -collineation  $\Phi$  is a collineation with center  $C$  and axis  $\alpha$ . If  $C \in \alpha$  then  $(C, \alpha)$ -collineation  $\Phi$  is called  $(C, \alpha)$ -elation (or elation).  $M$  is called  $(C, \alpha)$ -

transitive, if for all  $P, Q \in \mathcal{P}$ ,  $P, Q \square C$ ,  $P, Q, \square \alpha$ ,  $P, Q, C$  collinear, there is a  $(C, \alpha)$ -collineation mapping  $P$  to  $Q$ .

**Definition:** Let  $M$  be a PK-plane. If  $M$  is  $(C, \alpha)$ -transitive for all  $(C, \alpha)$ ,  $C \in \alpha$ , then  $M$  is called a Moufang-Klingenberg plane (MK-plane).

If  $M$  is MK-plane the canonic image  $M^*$  is a Moufang plane.

MK-planes are coordinatized with local alternative rings. In [2], MK-planes are coordinatized, using Duga's method (c.f. [8]). Before we are going to summarize their results we recall the definition of local alternative rings:

**Definition.** An alternative ring  $R$  with identity element  $1$  is called local, if the  $I$  of its non-units is an ideal.

It is shown ([2]) that for every local alternative ring there is a corresponding coordinate plane, which is an MK-plane, and conversely, that every MK-plane may be coordinatized by local alternative ring. Now, we are going to summarize their coordinatization (For more information about coordinatization of MK-planes see [2], [5], [14]):

Let  $M$  be a MK-plane and  $(O, E, U, V)$  a basis of  $M$  i.e. its canonic image  $(\Psi(O), \Psi(E), \Psi(U), \Psi(V))$  is a non-degenerate quadrangle in  $M^*$ . Let  $\ell = OE$ ,  $W: = \ell \cap UV$ ,  $R: = \{P \in \mathcal{P} \mid P \in \ell, P \square W\}$ ,  $I: = \{P \in R \mid P \square O\}$  especially  $o: = O$ ,  $1: = E$ , Let  $g_\infty: = UV$  and if  $P \in \ell$ ,  $P \square g_\infty$  then, when  $x \in R$ ,  $(x, x, 1)$  is taken as the coordinates of  $P$ . The points  $P \in \mathcal{P}$  of  $M$  get their coordinates as follows:

- (i)  $P \square g_\infty \Rightarrow P = (x, y, 1)$  where  $(x, x, 1) = PV \cap \ell$ ,  $(y, y, 1) = PU \cap \ell$ .
- (ii)  $P \square g_\infty, P \square V \Rightarrow P = (1, y, z)$  where  $(1, z, 1) = (PV \cap UE) \cap EV$ ,  $(1, y, 1) = OP \cap EV$ .
- (iii)  $P \square V \Rightarrow P = (w, 1, z)$  where  $(1, 1, z) = PU \cap I$ ,  $(w, 1, 1) = OP \cap EU$  where  $w, z \in I$ .

And lines  $g \in \mathcal{L}$  of  $M$  get their coordinates as follows:

- (i)  $g \square V \Rightarrow g = [m, 1, p]$  where  $(1, m, 1) = (g \cap g_\infty) \cap EV$ ,  $(o, p, 1) = g \cap OV$ .
- (ii)  $g \square V, g \square g_\infty \Rightarrow g = [1, n, p]$  where  $(n, 1, 1) = (g \cap g_\infty) \cap EU$ ,  $(p, 0, 1) = g \cap OU$ .
- (iii)  $g \square g_\infty \Rightarrow g = [q, n, 1]$  where  $(1, o, q) = g \cap OU$ ,  $(o, 1, n) = g \cap OV$

where  $q, n \in I$ .

Now we are going to give the algebraic correspondings of the incidence relation:

$$(x, y, l) \in [m, l, p] \Leftrightarrow y = xm + p$$

$$(l, y, z) \in [m, l, p] \Leftrightarrow y = m + zp$$

$$(w, l, z) \notin [m, l, p]$$

$$(x, y, z) \in [l, n, p] \Leftrightarrow x = yn + p$$

$$(w, l, z) \in [l, n, p] \Leftrightarrow w = n + zp$$

$$(l, y, z) \notin [l, n, p]$$

$$(l, y, z) \in [q, n, l] \Leftrightarrow z = q + yn$$

$$(w, l, z) \in [q, n, l] \Leftrightarrow z = wq + n$$

$$(x, y, l) \notin [q, n, l].$$

Therefore, the lines may be given as follows with the incidence points:

$$[m, l, p] = \{(x, xm + p, l) \mid x \in R\} \cup \{(l, zp + m, z) \mid z \in I\}$$

$$[l, n, p] = \{(yn + p, y, l) \mid y \in R\} \cup \{(zp + n, l, z) \mid z \in I\}$$

$$[q, n, l] = \{(l, y, yn + q) \mid y \in R\} \cup \{(w, l, wq + n) \mid w \in I\}.$$

Now we can state;

**Theorem 1.** (cf. [2]) Let  $M$  be a MK-plane coordinatized as above. Then  $(R, +, \cdot)$  is a local alternative ring with  $I$  the ideal of nonunits. Neighbourhood in  $M$  is characterized by

$$(x_1, x_2, x_3) \square (y_1, y_2, y_3) \Leftrightarrow x_i - y_i \in I, i = 1, 2, 3$$

$$[x_1, x_2, x_3] \square [y_1, y_2, y_3] \Leftrightarrow x_i - y_i \in I, i = 1, 2, 3$$

Conversely, given a local alternative ring  $R$  one can construct an MK-plane  $M(R)$  over  $R$ .

### 3. CONJUGACY AND CROSS-RATIO IN LOCAL ALTERNATIVE RINGS

Let  $A$  be an alternative ring with identity element. On the  $R: = R(\varepsilon) = A + A\varepsilon$ , ( $\varepsilon^2 = 0$ ) the operations  $(+)$  and  $(\cdot)$  may be defined as follows:

$$(a_1 + a_2\varepsilon) + (b_1 + b_2\varepsilon) = a_1 + b_1 + (a_2 + b_2)\varepsilon$$

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon$$

Then, there is a local alternative ring  $I = A\varepsilon$  is ideal of non units (cf. [4]).

In this section, generalizing the well known ones e.g. [9] or [18], the definition of cross-ratio is given on the projective line which is coordinatized with the local alternative rings like as  $R$ .

In [9], the concept of cross-ratio over the projective line which is coordinatized by alternative division ring  $A$  is given as conjugacy class, i.e.

$$a \equiv b \Leftrightarrow \exists \text{ unit } c \text{ such that } a = c^{-1}bc.$$

Before the state the characterization of “ $\equiv$ ” on  $R$  we want to give some concepts which are given in [9] for  $A$ :

$A$  is a Cayley division algebra over its center  $Z$  which is equipped with an involutory anti-automorphism  $K: x \rightarrow \bar{x}$ . The norm form  $N: x \rightarrow N(x) = x\bar{x} (= \bar{x}x)$  is a quadratic form which is multiplicative in the sense that  $N(ab) = N(a)N(b)$  for all  $a, b \in A$ . The trace linear form on  $A$  is defined by  $T: x \rightarrow T(x) = x + \bar{x}$ . The trace form is symmetric ( $T(ab) = T(ba)$ ) and associative ( $T(a(bc)) = T((ab)c)$ ). Then the following characterization of conjugacy in proper alternative fields may be given as follows:

**Lemma 1.** Let  $A$  be non-associative alternative division ring. Then all  $a, b \in A$   $a \equiv b \Leftrightarrow N(a) = N(b)$  and  $T(a) = T(b)$ .

More information about the norm and trace form are given in [11] and [16].

By Lemma 1 “ $\equiv$ ” is an equivalence relation over every alternative division ring  $A$ .

The concepts of involutory anti-automorphism, norm and trace form are may be extended over  $R$  as follows:

Let  $x = x_1 + x_2\varepsilon \in R$ . The map

$$K: R \rightarrow R$$

$$x \rightarrow K(x) = \bar{x} = \bar{x}_1 + \bar{x}_2\varepsilon$$

is involutory  $Z(\varepsilon)$  ( $= Z + Z\varepsilon$ )-linear anti-automorphism over  $R$ . The norm form  $N(x) = x\bar{x} (= \bar{x}x)$  is multiplicative and trace form  $T(x) = x + \bar{x}$  is  $Z(\varepsilon)$ -linear.

**Lemma 2.** Let  $a, b, c \in R$ ,  $a = a_1 + a_2\varepsilon$ .

i) There is a symmetric bilinear form  $f$  such that

$$N(a) = N(a_1) + f(a_1, a_2)\varepsilon.$$

ii)  $T(a) = T(a_1) + T(a_2)\varepsilon$ .

iii) The trace form is symmetric and associative, i.e.

$$T(ab) = T(ba), \quad T(a(bc)) = T((ab)c).$$

iv)  $T(a) = T(\bar{a})$ ,  $N(a) = N(\bar{a})$ .

v)  $f(a_1, a_2) = T(a_1\bar{a}_2)$

vi)  $N(a) = 0 \Leftrightarrow a \in I$ . If  $a \in R/I$   $a^{-1} = N(a)^{-1} \bar{a}$ .

iv) Since addition is commutative and  $K$  is an involutory

$$T(a) = a + \bar{a} = \bar{a} + a = \bar{a} + \bar{\bar{a}} = T(\bar{a})$$

$$N(a) = a\bar{a} = \bar{a}a = \bar{\bar{a}\bar{a}} = N(\bar{a})$$

v)  $N(a) = a\bar{a} = (a_1 + a_2\varepsilon)(\bar{a}_1 + \bar{a}_2\varepsilon) = a_1\bar{a}_1 + (a_1\bar{a}_2 + a_2\bar{a}_1)\varepsilon$   
 $= N(a_1) + f(a_1, a_2)\varepsilon$

$$\Rightarrow f(a_1, a_2) = a_1\bar{a}_2 + a_2\bar{a}_1 = T(a_1\bar{a}_2) (= T(a_2\bar{a}_1))$$

(i), (ii), (iii) and (vi) are proved in [4].

Using together the properties  $T(ab) = T(ba)$  and  $T(a) = T(\bar{a})$  we obtain  $f(a_1, a_2) = T(a_1\bar{a}_2) = T(a_2\bar{a}_1) = T(\bar{a}_1a_2) = T(\bar{a}_2a_1)$ .

Now we can state the characterization of " $\equiv$ " over the  $R$  as follows:

**Theorem 2.** ([4]) Let  $R$  be non-associative,  $a = a_1 + a_2\varepsilon$ ,  $b = b_1 + b_2\varepsilon \in R$ .

(i) If  $a_1 \in Z$ , then  $a \equiv b \Leftrightarrow a_1 = b_1$  and  $a_2 \equiv b_2$

(ii) If  $a_1 \notin Z$ , then  $a \equiv b \Leftrightarrow N(a) = N(b)$  and  $T(a) = T(b)$

**Lemma 3.** " $\equiv$ " is an equivalence relation over  $R$ .

**Proof:** Let  $a, b, c \in R$ ,  $a = a_1 + a_2\varepsilon$ ,  $b = b_1 + b_2\varepsilon$ ,  $c = c_1 + c_2\varepsilon$ .

Because of the definition " $a \equiv a$ " and " $a \equiv b \Leftrightarrow b \equiv a$ " is obvious. Let  $a \equiv b$  and  $b \equiv c$ . Then there are two case:

i) If  $a_1 \in Z$ , then  $a_1 = b_1 \in Z$ ,  $b_1 = c_1$  i.e.  $a_1 = c_1$  and  $a_2 \equiv b_2$ ,  $b_2 \equiv c_2$  i.e.  $a_2 \equiv c_2$  (Lemma 1). Therefore  $a \equiv c$ .

ii) If  $a_1 \notin Z$  then we have  $N(a) = N(b)$ ,  $T(a) = T(b)$ ,  $N(b) = N(c)$ ,  $T(b) = T(c)$ . Therefore  $N(a) = N(c)$ ,  $T(a) = T(c)$ . Hence  $a \equiv c$ .

In [4], it is shown that for all  $a, b, c \in R$   $ab \equiv ba$  and  $a(bc) \equiv (ab)c$ .

An equivalence class of  $x \in R$  is denoted by  $[x]$ . If  $x \in A$ ,  $[x]_A$  is denoted its conjugacy class in  $A$ . It can be shown that, by direct computation,  $[x]_A = [x] \cap A$ .

#### 4. CROSS-RATIO OVER PROJECTIVE LINE

Let  $\infty \notin A$ . The projective line which is coordinatized by  $A$  is denoted by  $P(A)$ . Therefore  $P(A) = A \cup \{\infty\}$ . Generalizing this over  $R$ ,  $P(R) = R \cup I^{-1}$  is obtained, where  $I^{-1} := \{(a\varepsilon)^{-1} \mid a\varepsilon \in I\}$  consist of formal inverses of the non-units of  $R$ . On  $P(R)$ , neighbour relation "o" is defined by;

$$x \square y : \Leftrightarrow (x, y \in I^{-1}) \text{ or } (x, y \in R, x - y \in I).$$

Using the fact that  $I$  is an ideal, it can be shown that neighbouring is an equivalence relation.

By putting  $(0\varepsilon)^{-1} = 0^{-1} = \infty$ ,  $P(A) \subseteq P(R)$  is obtained. The mapping  $\Psi: P(R) \rightarrow P(A)$ ,  $\Psi(x_1 + x_2\varepsilon) = x_1$ ,  $\Psi((a\varepsilon)^{-1}) = \infty$  is surjective and leaves "o" invariant. Also  $xoy \Leftrightarrow \Psi(x) = \Psi(y)$ . So  $P(R)/\square \cong P(A)$  via the canonic epimorphism  $\Psi$ .

The operations with the elements of  $I^{-1}$  is as follows (cf. [4]):

Let  $(a\varepsilon)^{-1} \in I^{-1}$ ,  $c \in R$  and  $q \in R/I$ . Then,

$$(a\varepsilon)^{-1} + c : = (a\varepsilon)^{-1} =: c + (a\varepsilon)^{-1}$$

$$q(a\varepsilon)^{-1} : = (aq^{-1}\varepsilon)^{-1} (= a(q_1^{-1}\varepsilon)^{-1}); \quad q = q_1 + q_2\varepsilon$$

$$(a\varepsilon)^{-1}q : = (q^{-1}a\varepsilon)^{-1} (= (q_1^{-1}a\varepsilon)^{-1})$$

$$((a\varepsilon)^{-1})^{-1} = a\varepsilon$$

Other terms are not defined.

With the help of these rules, some special permutations (which are given in [9] over  $A$ ) can be given over  $R$ :

(1):  $t_\alpha: x \rightarrow x + \alpha ; \alpha \in R$

(2):  $l_q: x \rightarrow qx ; q \in R / I$

(3):  $r_q: x \rightarrow xq ; q \in R / I$

(4):  $i: x \rightarrow x^{-1}$

Let  $G$  be a group generated by all these permutations, i.e.  $G = \langle t_\alpha, l_q, r_q, i \rangle$   $a \in R, q \in R / I$ . Since  $r_q = i l_{q^{-1}} i$ ,  $G$  is generated by all  $t_\alpha, l_q, i$ .

**Lemma 4.** For the group  $G = G(R)$ , following statements are satisfied:

i)  $G$  preserves neighbourhood.

ii)  $G$  acts transitively on triples of pairwise nonneighbouring points of  $P(R)$ .

**Proof:** i) is easy to show that  $t_\alpha, l_q$  and  $i$  preserves neighbourhood.

ii) It is sufficient to see that there is a  $\sigma \in G$  such that  $\sigma(x) = 0, \sigma(y) = 1$  and  $\sigma(z) = \infty$ , for all pairwise non-neighbouring  $x, y, z \in P(R)$ . If  $\sigma$  is defined as follows then proof will be completed.

$$\sigma = \begin{cases} r_{(y-x)^{-1}} t_{-x} & ; \text{if } z = \infty := (0\varepsilon)^{-1} \\ r_{(y^{-1}-x^{-1})} t_{-x^{-1}} i & ; \text{if } z = 0 \\ r_{((y-z)^{-1}-(x-z)^{-1})} t_{-(x-z)^{-1}} i t_{-z} & ; \text{if } z \neq \infty, z \neq 0 \end{cases}$$

**Definition.** Let  $a, b, c, d \in P(R)$  be pairwise non-neighbouring. The cross-ratio  $(a, b; c, d)$  of the elements  $a, b, c, d$  is defined as a conjugacy class via:

$(a, b; c, d) = [((a-d)^{-1} (b-d)) ((b-c)^{-1} (a-c))]$ ; if  $a, b, c, d \in R$

$(s^{-1}, b; c, d) = [((1 + ds) (b-d)) ((b-c)^{-1} (1-cs))]$ ; if  $s^{-1} \in I^{-1}, b, c, d \in R$

$(a, s^{-1}; c, d) = [((a-d)^{-1} (1-ds)) ((1 + cs) (a-c))]$ ; if  $s^{-1} \in I^{-1}, a, c, d \in R$

$(a, b; s^{-1}, d) = [((a-d)^{-1} (b-d)) ((1 + sb) (1-sa))]$ ; if  $s^{-1} \in I^{-1}, a, b, d \in R$

$(a, b; c, s^{-1}) = [((1 + sa) (1-sb)) ((b-c)^{-1} (a-c))]$ ; if  $s^{-1} \in I^{-1}, a, b, c \in R$

The theorem which is given in [9] as Theorem 2 over alternative division ring  $A$  can be generalized over  $R$  i.e. every cross-ratio consist onlt of elements of  $(R \setminus (\{0, 1\} + A\varepsilon))$ . Conversely, the conjugacy class of any such element appears as a crossratio: Given three pairwise non-



neighbouring points  $a, b, c$  an element  $r \in R / (\{0, 1\} + A\epsilon)$ , then there is a point  $d \notin a, b, c$  with  $[r] = (a, b; c, d)$  and if  $r \in Z(\epsilon)$  then  $d$  is unique (cf. [4]).

A permutation  $\Phi$  of  $P(R)$  preserves cross ratios, if it preserves neighbourhood, and if

$$(a, b; c, d) = (\Phi(a), \Phi(b); \Phi(c), \Phi(d))$$

holds for all pairwise non-neighbouring  $a, b, c, d \in P(R)$ . The group of all such  $\Phi$  is denoted by  $S(R) = S$ .

Now, we can state a theorem which is expressed cross-ratio in terms of addition, subtraction and inversion:

**Theorem 3.** Let  $a, b, c, d \in R$  non-neighbouring pairwise points. Then

$$(a, b; c, d) = [((a-b)^{-1} - (a-d)^{-1}) ((a-b)^{-1} - (a-c)^{-1})^{-1}].$$

**Proof:** By Lemma 1, the multiplicative of  $N$ , and associativity of  $T$   $u' = ((a-d)^{-1} (b-d)) ((b-c)^{-1} (a-c))$  is conjugate to  $u = (((a-d)^{-1} (b-d)) (b-c)^{-1} (a-c))$ .  $\Rightarrow$

$u = (((a-d)^{-1} (b-d)) (b-c)^{-1} (a-c)) \Rightarrow (u(a-c)^{-1}) (b-c) = (a-d)^{-1} (b-d)$ . Thus,

$$((a-d)^{-1} (b-d)) (a-b)^{-1} = ((u(a-c)^{-1}) (b-c)) (a-b)^{-1} \quad (*)$$

First we compute the left hand side:

$$\begin{aligned} ((a-d)^{-1} (b-d)) (a-b)^{-1} &= ((a-d)^{-1} ((a-d) - (a-b))) (a-b)^{-1} \\ &= ((a-d)^{-1} (a-d) - (a-d)^{-1} (a-b)) (a-b)^{-1} \\ &= (1 - (a-d)^{-1} (a-b)) (a-b)^{-1} \\ &= (a-b)^{-1} - (a-d)^{-1} ((a-b) (a-b)^{-1}) \\ &= (a-b)^{-1} - (a-d)^{-1}. \end{aligned}$$

Putting this result in (\*):

$$\begin{aligned} (a-b)^{-1} - (a-d)^{-1} &= ((u(a-c)^{-1}) (b-c)) (a-b)^{-1} \\ \Rightarrow (((a-b)^{-1} - (a-d)^{-1}) ((a-b)) (b-c)^{-1} (a-c)) &= u \\ \Rightarrow [(((a-b)^{-1} - (a-d)^{-1}) (a-b)) (b-c)^{-1} (a-c)] &= [u] \\ \Rightarrow [((a-b)^{-1} - (a-d)^{-1}) ((a-b)) ((b-c)^{-1} (a-c))] &= [u] \end{aligned}$$

$$\begin{aligned} &\Rightarrow [((a-b)^{-1}(a-d)^{-1}) (((a-c)^{-1}((a-c)-(a-b))) (a-b)^{-1})^{-1}] = [u] \\ &\Rightarrow [((a-b)^{-1}(a-d)^{-1}) ((a-b)^{-1}((a-c)^{-1}(a-b)) (a-b)^{-1})^{-1}] = [u] \\ &\Rightarrow [((a-b)^{-1}(a-d)^{-1}) ((a-b)^{-1}(a-c)^{-1})^{-1}] = [u] = [u']. \end{aligned}$$

In fact, this proof is given in [9] by using to the idea; "Moufang identities remain valid when one  $x$  is replaced by its inverse.". But it is not true (cf. [6], [7]). For this, we gave a formal proof above.

**Lemma 5.** For all  $a, b, c, d \in P(R)$  pairwise non-neighbouring elements  $(a, b; c, d) = (b, a; d, c)$ .

**Proof:** There are three cases:

(i) If  $a, b, c, d \in R$ ;

$$\begin{aligned} (a, b; c, d) &= [((a-d)^{-1}(b-d)) ((b-c)^{-1}(a-c))] \\ &= [((b-c)^{-1}(a-c)) ((a-d)^{-1}(b-d))] \\ &= [b, a; d, c] \end{aligned}$$

(ii) If  $s^{-1} \in I^{-1}$ ,  $b, c, d \in R$ ;

$$\begin{aligned} (s^{-1}, b; c, d) &= [((1+ds)(b-d)) ((b-c)^{-1}(1-cs))] \\ &= [((b-c)^{-1}(1-cs)) (((1+ds)(b-d))] \\ &= [b, s^{-1}; d, c] \end{aligned}$$

(iii) If  $s^{-1} \in I^{-1}$ ,  $a, b, c \in R$  it is shown by similar way.

**Theorem 4.**  $G$  is a subgroup of  $S$ .

To prove this it is sufficient to see all  $t_\alpha, r_\alpha, i \in S$  i.e. to see  $t_\alpha, r_\alpha$  and  $i$  preserves cross-ratio. If  $a, b, c, d \in R/I$ ,  $t$  is proved in [9]. By Lemma 5 it must be seen only to the cases  $a \in I^{-1}$  and  $d \in I^{-1}$ . These are may be seen by easy computation and some of these computations are given in [4].

**Lemma 6.** Let  $a, b, c, d \in P(R)$  be pairwise non-neighbouring and  $[x]^{-1}$  is defined to be  $[x^{-1}]$  and  $1-[x] = [1-x]$  then

$$\begin{aligned} (a, b; c, d)^{-1} &= (b, a; c, d) \\ 1-(a, b; c, d) &= (a, c; b, d) \\ (a, b; c, d) &= (b, a; d, c) = (c, d; a, b) = (d, c; b, a) \end{aligned}$$

**Proof:** By lemma 4-ii and Theorem 4 we can chose  $a = \infty, b = 0, c = 1$ . Then this lemma is proved by easy computation.

Corollary. ([4]) Let  $(a, b, c, d)$  and  $(x, y, z, t)$  are pairwise non-neighbouring quadruples of  $P(R)$ . Then

$$(a, b; c, d) = (x, y; z, t) \Leftrightarrow \exists \gamma \in G \ni \gamma(a) = x, \gamma(b) = y, \gamma(c) = z, \gamma(d) = t.$$

### 5. PERMUTATION GROUPS ON A LINE IN MK-PLANES.

Let  $M$  be a MK-plane which is coordinatized with respect to  $(O, E, U, V)$ ,  $R$  the corresponding local alternative ring,  $I$  the ideal of its non-units. Consider a line

$$g = OV = [1, 0, 0] = \{(0, y, 1) \mid y \in R\} \cup \{(0, 1, z) \mid z \in I\}$$

Let  $P(R) = PUI^{-1}$ ,  $I^{-1} = \{z^{-1} \mid z \in I\}$ .  $P(R)$  is identified with  $g$  as follows:

$$y \longleftrightarrow (0, y, 1), \quad z^{-1} \longleftrightarrow (0, 1, z).$$

**Definition.** Let  $h, k \in \mathcal{L}$ ,  $C \in \mathcal{P}$ ,  $C \not\sqsupset h, k$ . Then well-defined bijection

$$\sigma := \sigma(h, C, k): h \longrightarrow k \ni \sigma(X) = \bar{X}C \cap k$$

mapping  $h$  to  $k$  is called a perspectivity from  $h$  to  $k$  with centre  $C$ . A finite product of perspectivities is called a projectivity. The set of all projectivities mapping an any line  $g$  onto itself is a group which is denoted by  $\mathcal{K}$  or  $\mathcal{K}(g)$ .

The following lemma is given in [5] like as [1]:

**Lemma 7.** The group  $\mathcal{K}$  preserves the neighbour relation and transitively on the triples of pairwise non-neighbouring point of  $P(R)$ . Moreover,  $\mathcal{K}$  is generated by the products of three perspectivities.

**Definition 5.** (cf. [5]) For every MK-planes  $G \leq \mathcal{K}$ .

### 6. ALGEBRAIC DESCRIPTION OF THE PROJECTIVITIES

In this section we are going to summarise some definitions and theorems which are given in [5].

**Lemma 8.** If  $M$  is lineary cohesive then every perspectivity is induced by an elation, and every projectivity is induced by a projective collineation.

Now we shall consider coordinatization of  $M$  with respect to arbitrary bases  $(O', E', U', V')$ . Without lost of generality, we may always

take the set  $R$  as the set of coordinates, and subset  $I$  as the coordinates of points neighbouring  $O'$ . Of course,  $R$  is again a local alternative ring,  $I$  the set of its non-units. If  $M$  is coordinatized with respect to the basis  $(O, E', U, V)$  where  $1 = OV \cap UE = OV \cap UE'$ , then for the points  $y \in g = P(R)$  with  $y \in R$ , we can always assume that the condition

$$(0, y, 1) = y = (0, y, 1)'$$

holds.

**Lemma 9.** Let  $R' = (R, \oplus, o)$  be the coordinate alternative ring of  $M$  with respect to the basis  $(O, E', U, V)$  where  $g \cap UE' = 1 = g \cap UE$ . Then there is a  $q \in R/I$ , such that  $a \oplus b = a + b$ ,  $aob = (aq)(q^{-1}b)$  hold for all  $a, b \in R$ .

By using the aboves the following theorem can be proved:

**Theorem 6.** Let  $M$  be a cohesive MK-plane such that the canonic image  $M^*$  is different from the smallest projective plane. Then  $\mathcal{K} = G$ .

## ÖZET

Bu çalışmada [3], [4] ve [9] da verilen çifte oran tanımları ışığında konunun geniş bir incelemesi yapılmış ve [9] daki bazı sonuçların hatalı bir cebirsel ifadeye dayalı olan ispatları yerine formal ispatları verilmiştir. Ayrıca hesaplamalarda çok kullanışlı olan bazı temel sonuçlar çıkarılmıştır.

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