Commun. Fac. Sci. Univ. Ank. Series A_1
V. 41. pp. 209–214 (1992)

ON THE CONJUGACY CLASSES OF p^2: $GL_2(p)$ - "p ODD PRIME"

M.I.M. AL ALİ

Mu'tah University.

(Received Dec. 6, 1990; Accepted July 14, 1992)

INTRODUCTION

A particular procedure has been followed in [1] to construct the conjugacy classes of the split extension p^2: $GL_2(p)$. The object of this paper is to develop a general method for constructing the conjugacy classes of p^2: $GL_2(p)$, where p^2 is an elementary Abelian p–group of order p^2. This procedure can be used to construct the conjugacy classes of the split extension p^n: k where k is any finite group. A brief description of the character table of p^2: $GL_2(p)$ is also given, the character table of p^2: $GL_2(p)$ plays a big role in the construction of the character table of the maximal subgroup p^{1+2}: $GL_2(p)$ of the projective symplectic group $PSP_4(p)$ p–prime [4], where p^{1+2} is the extra special group of order p^3, this is because $(p^{1+2}: GL_2(p)) / Z(p^{1+2}) \simeq p^2: GL_2(p)$ where $Z(p^{1+2})$ is the center of p^{1+2} in p^{1+2}: $GL_2(p)$. These character tables are of great importance to the documentation programme of finite simple groups [3].

1. The Conjugacy Classes of $GL_2(p)$

The conjugacy classes of $GL_2(p)$ has been taken from Steinberg paper [5], and they are presented below. Let ρ and σ be a primitive element of $GF(p)^*$ and $GF(p^2)^*$ respectively such that $\rho = \sigma^{q+1}$, where $GF(p)^* = GF(p) \setminus \{0\}$.

2. The Conjugacy Classes of p^2: $GL_2(p)$

Denote p^2: $GL_2(p)$ by H:K, to find the conjugacy classes of the split extension H: K, we need to find the conjugacy classes of a general element (h, k). Two elemants (h_1, k_1) and (h_2, k_2) cannot be conjugate if $(1, k_1)$ is not conjugate to $(1, k_2)$. We can assume that $k_1 = k_2$. Then in order to see whether (h_1, k_1) and (h_2, k_1) are conjugate, we need only conjugate by elements (x, y) such that:

Family	Element	Number of Classes	Number of Elements in each Class
A_1	$\begin{pmatrix} \rho^a & \\ & \rho^a \end{pmatrix}$	p–1	1
A_2	$\begin{pmatrix} \rho^a & \\ 1 & \rho^a \end{pmatrix}$	p–1	p^2-1
A_3	$\begin{pmatrix} \rho^a & \\ & \rho^b \end{pmatrix}_{a \neq b}$	$\frac{1}{2}$ (p–1) (p–2)	p (p + 1)
B	$\begin{pmatrix} \sigma^a & \\ & \sigma^b \end{pmatrix}$ $a \neq$ mult $(p+1)$ $b \neq ap \bmod (p^2-1)$	$\frac{1}{2}$ p (p–1)	p (p–1)

$$(x, y)\,(h_1, k_1)\,(x, y)^{-1} = (h_2, k_1).$$

This means that (h_1, k_1) is conjugate to (h_2, k_1) if $(x, y)\,(h_1, k_1)\,(x, y)^{-1} = (h_2, k_1)$, for some (x, y), and also this means that (h_1, k_1) is conjugate to (h_2, k_1) if and only if (h_2, k_1) lies in the orbit of (h_1, k_1) under the set of all elements (x, y) such that $(x, y)\,(h, k_1)\,(x, y)^{-1} = (h', k_1)$, where $h, h' \in H$ (i.e. stabilizer of the coset $\{(h, k_1) \mid h \in H\}$). Clearly $\{(h, 1)\}$ lies in the stabilizer of $\{(h, k_1)\}$. Since

$$(h, 1)\,(h', k_1)\,(h^{-1}, 1) = (h, 1)\,(h'\,h^{-1}, k_1) = (hh'\,h^{-1}, k_1),$$

where $hh'h^{-1}$ might not be h' (if H is not Abelian), H is contained in stabilizer of $\{(h, k_1) \mid h \in H\}$.

Also $(h, x)\,(h', k_1)\,(h, x)^{-1} = (*, xk_1x^{-1}) = (*, k_1)$ if and only if $(1, x) \in C_K(k_1)$, and so the stabilizer of the coset $\{(h, k_1)\}$ is $H{:}\,C_K(k_1)$, where $C_K(k_1)$, is the centralizer of k_1 in K. Note that H is a subgroup of $H{:}\,C_K(k_1)$, so the orbits of H acting on the coset $\{(h, k_1)\}$ are blocks of imprimitivity.

The elementary Abelian p–group H can be considered as a 2–dimensional vector space $v_2(p)$ over GF(p). Let $k \in K$ be a representative of the conjugacy class $\hat{k}$. The classes of H: K which lie below k are of the form hk for some h's $\in$ H. The action of K on H,

$$h \xrightarrow{k} h^k = k^{-1}hk$$

can be identified with

$$\underline{u} \xrightarrow{k} \underline{u}\, k$$

where $\underline{u}$ is the 2–tuple which corresponds to h with respect to the basis $A = \{(1, 0), (0, 1)\}$ of $V_2(p)$, and the element hk can be represented by 3 x 3 matrix

$$\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & \\ & k \\ 0 & \end{array}\right]$$

Because if $k_1, k_2 \in K = GL_2(p)$ and $\underline{u}_1, \underline{u}_2$ are the two 2–tuples which correspond to $h_1. h_2 \in H$, respectively, we have

$$\left[\begin{array}{c|c} 1 & \underline{u}_1 \\ \hline 0 & \\ & k_1 \\ 0 & \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{u}_2 \\ \hline 0 & \\ & k_2 \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} 1 & \underline{u}_1 k_2 + \underline{u}_2 \\ \hline 0 & \\ & k_1 k_2 \\ 0 & \end{array}\right]$$

which corresponds to $(h_1, k_1)\,(h_2, k_2) = (h_1^{k_2} + h_2, k_1\, k_2)$.

Now we give a general description for the construction of the conjugacy classes of H:K.

Choose an element $(h^*, k) \in H: K$, this element can be identified with,

$$\left[\begin{array}{c|c} 1 & \underline{u}^* \\ \hline 0 & \\ & k \\ 0 & \end{array}\right]$$

where u^* is the 2–tuple corresponding to h^* with respect to the basis A, then we have

$$\left[\begin{array}{c|c} 1 & \underline{u}_1 \\ \hline 0 & \\ & I \\ 0 & \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{u}^* \\ \hline 0 & \\ & k \\ 0 & \end{array}\right] \left[\begin{array}{c|c} 1 & -\underline{u}_1 \\ \hline 0 & \\ & k \\ 0 & \end{array}\right] =$$

$$\left[\begin{array}{c|c} 1 & \underline{u}_1 k + \underline{u}^* \\ \hline 0 & \\ & k \\ 0 & \end{array}\right] \left[\begin{array}{c|c} 1 & -\underline{u}_1 \\ \hline 0 & \\ & I \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} 1 & \underline{u}_1 k + \underline{u}^* - \underline{u}_1 \\ \hline 0 & \\ & k \\ 0 & \end{array}\right].$$

This multiplication can be abbreviated to

$$(\underline{u}_1, I)\,(\underline{u}^*, k)\,(-\underline{u}_1, I) = (\underline{u}_1 k + \underline{u}^* - \underline{u}_1, k).$$

We first determine the length of the block of imprimitivity containing $(\underline{u}^*, \underline{k})$ by considering expressions of the form

$$(\,(ru_{11} + u^*_1 - u_{11} + tu_{21},\ su_{11} + u_2^* - u_{21} + vu_{21}),\ k) \text{ where}$$

$$\underline{u}_1 = (u_{11}, u_{21}) \quad k = \begin{pmatrix} r & s \\ t & v \end{pmatrix}$$

and $u^* = (u^*_1, u^*_2)$. Suppose that $r = 1$ and $t = 0$, this means that we get an orbit of length p, so we start with a given element $(\underline{0}, k)$ and determine the p–elements in the orbit under the action of k. Next, we want to see whether this orbit joins with another orbit. Since $C_K(k)$ acts on these orbits (i.e. moves them setwise among each other), we get an element $k^* \in C_K(k)$ and compute

$$\begin{aligned}(\underline{0}, k^*)\ &(\,(u^*_1, su_{11} + u^*_2 - u_{21} + vu_{21}), k)\,(\underline{0}, k^{*-1}) \\ &= (\,(u^*_1, su_{11} + u^*_2 - u_{21} + vu_{21}), k^*k)\,(0, k^{*-1}) \\ &= (\,(u^*_1, su_{11} + u^*_2 - u_{21} + vu_{21})\,k^{*-1}, k)\end{aligned}$$

and check to see whether

$$((u^*_1, su_{11} + u^*_2 - u_{21} + vu_{21})\,k^{*-1}, k) \in \{((u_1^*, su_{11} + u_2^* - u_{21} + vu_{21}), k)\}.$$

If it is not then the orbit increases to twice its length. Continuing in this way we eventually obtain the orbit length.

Now we given some examples to show how the classes of H: K were found. The same procedure is applied for the others.

Example (1)

Let $\begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix} \in A_1$, then $(\,(u_{11}, u_{21}), 1)\,(\,(u_1^*, u_2^*), \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix})$

$$(\,(-u_{11}, -u_{21}), 1) = (\,(u^*_1, u_{21}\,\rho^a + u^*_2 - u_{21}), \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix})$$

if we let $u_1^* = u^*_2 = 0$, we find an orbit of length p, namely

$((0, u_{21}\,\rho^a - u_{21}), \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix})$, then if we conjugate by $\begin{pmatrix} l & \\ & m \end{pmatrix} \in C_K\begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}$ we get

$\left((0,*), \begin{pmatrix}1 & \\ & \rho^a\end{pmatrix}\right)$ which is the same orbit. Now if $\underline{u}^* = (u_1^*, u_2^*) \neq \underline{0}$ and if we conjugate $\left((u_1^*, u_{21}\,\rho^a + u_2^* - u_{21}), \left(\begin{pmatrix}1 & \\ & \rho^a\end{pmatrix}\right)\right.$ by $\begin{pmatrix}l & \\ & m\end{pmatrix}$ we get an orbit of the form $\left((l^{-1} u_1^*, *), \begin{pmatrix}1 & \\ & \rho^a\end{pmatrix}\right)$ of length p(p−1), this means that we have two conjugacy classes of H:K lie below $\begin{pmatrix}1 & \\ & \rho^a\end{pmatrix}$; their representatives are $\left(\underline{0}, \begin{pmatrix}1 & \\ & \rho^a\end{pmatrix}\right)$ and $\left((\underline{u}^*, \begin{pmatrix}1 & \\ & \rho^a\end{pmatrix}\right)$, $\underline{u}^* \neq 0$ and the order of these classes are p, p (p−1) respectively. The other conjugacy classes of K were treated in a similar manner. The complete results are given in the following table.

Class representative	$\left(\underline{0}, \begin{pmatrix}1 & \\ & 1\end{pmatrix}\right)$	$\left(\underline{u}, \begin{pmatrix}1 & \\ & 1\end{pmatrix}\right)_{\underline{u}\neq 0}$	$\left(\underline{0}, \begin{pmatrix}1 & \\ & \rho^a\end{pmatrix}\right)_{a\neq p-1}$
Number of classes	1	1	p−2
Orbit length	1	p^2-1	p
Centralizer	$p^2(p^2-1)(p^2-p)$	$p^2(p^2-p)$	$p(p^2-1)(p^2-p)$

Class representative	$\left(\underline{u}, \begin{pmatrix}1 & \\ & \rho_a\end{pmatrix}\right)$ $a\neq p-1$, $u\neq 0$	$\left(\underline{0}, \begin{pmatrix}1 & \\ 1 & 1\end{pmatrix}\right)$	$\left(u, \begin{pmatrix}1 & \\ 1 & 1\end{pmatrix}\right)_{u\neq 0}$	$\left(\underline{0}, \begin{pmatrix}\rho^a & \\ & \rho^a\end{pmatrix}\right)_{a\neq p-1}$
Number of classes	p−2	1	1	p−2
Orbit length	p (p−1)	p	p (p−1)	p^2
Centralizer	$(p+1)(p^2-p)$	$p(p^2-1)(p^2-p)$	$p(p^2-1)(p^2-p)$	$(p^2-1)(p^2-p)$

Class Representative	$\left(\underline{0}, \begin{pmatrix}\rho^a & \\ & \rho^a\end{pmatrix}\right)_{a\neq p-1}$	$\left(\underline{0}, \begin{pmatrix}\rho^a & \\ & \rho^b\end{pmatrix}\right)_{a\neq b}$	$\left(0, \begin{pmatrix}\sigma^a & \\ & \sigma^b\end{pmatrix}\right)_{a\neq \text{mult}(p+1)}$ $b\neq a p \bmod (p^2-1)$
Number of classes	p−2	$\frac{(p-2)(p-3)}{2}$	$\frac{1}{2}$ (p) (p−1)
Orbit length	p^2	p^2	p^2
Centralizer	$(p^2-1)(p^2-p)$	$(p^2-1)(p^2-p)$	$(p^2-1)(p^2-p)$

The total number of the conjugacy classes of p^2: $GL_2(p)$ is $p^2 + p-1$. The character table of p^2: $GL_2(p)$ can be constructed as follows. We extend the whole character table of $GL_2(p)$ to p^2: $GL_2(p)$. The character table of $GL_2(p)$ has been taken from [5] and presented below. Next we induce the 1–representations of $GL_2(p)$ to p^2: $GL_2(p)$. The extension gives p^2-1 irreducible characters of p^2: $GL_2(p)$ and the induction gives p–1 irreducible characters. The tensor product of one of these p–1 irreducible characters with an irreducible character of p^2: $GL_2(p)$ of degree p–1 completes the character table of p^2: $GL_n(p)$.

Note: The extension, induction and tensor product of characters can be easily handled using Clifford Programme [2].

Characters of $GL_2(p)$

In this table, $\chi_p{}^r$ for example, will denote a character of degree p. The superscript being used to distinguish between two characters of the same degree.

	χ_1^{n}	$\chi_p^{(n)}$	$\chi_{p+1}^{(m,n)}$	$\chi_{p-1}^{(n)}$
Element	$n = 1,2,\ldots,p-1$ $\varepsilon^{p-1} = 1$	$n = 1,2,\ldots,p-1$ $\varepsilon^{p-1} = 1$	$m,n = 1,2,\ldots,p-1$ $m \neq n$; $(m,n) = (n,m)$ $\varepsilon^{p-1} = 1$	$n = 1,2,\ldots,p^2-1$ $n \neq$ mult $(p+1)$ $\varepsilon^{p^2-1} = 1$
A_1	ε^{2na}	$p\varepsilon^{2na}$	$(p+1)\,\varepsilon^{(m+n)a}$	$(p-1)\,\varepsilon^{na(p+1)}$
A_2	ε^{2na}	0	$\varepsilon^{(m+n)a}$	$-\varepsilon^{na(p+1)}$
A_3	$\varepsilon^{n(a+b)}$	$\varepsilon^{n(a+b)}$	$\varepsilon^{ma+nb} + \varepsilon^{na+mb}$	0
B_1	ε^{na}	$-\varepsilon^{na}$	0	$-(\varepsilon^{na} + \varepsilon^{np})$

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