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ON THE BLASCHKE INVARIANTS OF THE PAIR OF THE GENERALIZED RULED SURFACES UNDER THE HOMOTHETIC MOTIONS

A.I. SIVRIDAG - S. KELEŞ

Faculty of Arts and Sciences,

Inönü University, MALATYA

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In this paper we discuss the relations between the Blaschke invariants of the pair of the moving and fixed (k + 1)–ruled surfaces $\overline{\Phi}$ and Φ under the homotheric motion in E^n. Moreover we find the relations for the pair of $\overline{\Psi}' \subset \overline{\Phi}$ and $\Psi' \subset \Phi$. As a special case of these relations we obtained some relations between the principal Blaschke invariants.

1. Homothetic Motions Of E^n And Blasehke Invarlants

A homothetic motion of E^n is described in matrix notation [4] by

$$x = S\bar{x} + C, \; S = hA, \; AA^T = I \tag{1}$$

where A^T is the transpose of the orthogonal matrix A, h is a real scalar matrix and

$$A: J \rightarrow O(n), \; C: J \rightarrow IR^n, \; h: J \rightarrow IR \tag{2}$$

are functions of differentiability class of C^r on a real interval J, and $\bar{x}$ and x are corresponding position vectors of the same point with respect to the rectangular co–ordinate systems of the so–called moving space $\bar{E}$ and fixed space E, respectively. At the initial time $t = t_0$ we assume that the co–ordinate systems of $\bar{E}$ and E coincides. To avoid the case of general affine transformations we assume that $h = h(t) \neq$ constant and to avoid the case of pure translations and pure rotations we also assume that

$$\dot{h}A + h\dot{A} \neq 0, \; \dot{C} \neq 0 \tag{3}$$

where (.) indicates d / dt.

Let $\bar{x}$ be fixed in $\bar{E}$ then (1) defines a parametrized curve, by (2), in E which we call it trajectory curve of $\bar{x}$ under the motion. Sinee we havs $\dot{\bar{x}} = O$, differentiating (1) we get the (trajectory) velocity vector $\dot{x}$, at the path–point x, in the form

$$\dot{x} = B\,(x-C) + \dot{C}, \quad B = \dot{S}S^{-1} \,. \tag{4}$$

Since the matrix $\dot{S}$ is regular matrix, (see [3]), $|B|$ doesn't vanish on J. So, for each $t \in J$ we get exactly one solution y(t) of the equation

$$B\,(t)\,(y-C(t)) + \dot{C}\,(t) = O \tag{5}$$

y(t) is the center of the instantaneous rotation of the motion at $t \in J$ and it is called the pole of the motion at $t \in J$. At a pole the velocity vector vanishes by the equation (4). Since $|B|$ does not vanish on J, by considering the regularity condition of the motion we get a differentiable curve $y: J \to E$ of poles in the fixed space E, called the fixed pole curve. By(1)' we can determin the moving pole curve $\bar{y}:J \to \bar{E}$ from the fixed pole curve point to point on J:

$$y(t) = S(t)\,\bar{y}(t) + C\,(t). \tag{6}$$

Let $\bar{y} \subset \bar{E}$ and $y \subset E$ be the moving and the fixed pole curves under the homothetic motion given by (1), respectively. Suppose that $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_k\}$ is an orthonormal vector field system at $\bar{y}(t)$ and $\bar{E}_k(t) = Sp\,\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_k\}$. Then $\bar{E}_k(t)$ generates a $(k+1)$–dimensional ruled surface with the leading curve y in the moving space $\bar{E}$ which is called the moving ruled sirface and it is denoted by $\bar{\Phi}$. $\bar{\Phi}$ has the following parameter representation on the interval J:

$$\bar{z}(t, \bar{u}_1, \ldots, \bar{u}_k) = \bar{y}(t) + \sum_{\nu=1}^{k} \bar{u}_\nu \bar{e}_\nu(t). \quad \bar{u}_\nu \in IR, \; t \in J. \tag{7}$$

Let $\{\varepsilon_1(t), \ldots, \varepsilon_k(t)\}$ be an orthogonal vector field system satisfying the following equations (8) and (9) at the point y(t) in the fixed space E:

$$S(\bar{e}_\nu) = \bar{\epsilon}_\nu \tag{8}$$

and

$$(\dot{A}A^{-1})\,\epsilon_\nu = O, \; 1 \leq \nu \leq k. \tag{9}$$

If we set

$$\epsilon_\nu = h e_\nu \;\; 1 \leq \nu \leq k \tag{10}$$

then we get the orthonormal vector field system $\{e_1, \ldots, e_k\}$. If $S_p\{e_1, \ldots, e_k\}$ is denoted by $E_k(t)$ then $E_k(t)$ generates a $(k+1)$–ruled surface with the leading curve y by (1) in the fixed space E. Under a homothetic motion such a ruled surface corresponds to the moving ruled surface $\bar{\Phi}$. This ruled surface is called the fixed ruled surface and denoted by Φ. The fixed ruled surface has the following parameter respresentation on the interval J:

$$Z(t, u_1, \dots, u_k) = y(t) + \sum_{\nu=1}^{k} u_\nu e_\nu,\ u_\nu \varepsilon R,\ t \varepsilon J. \tag{11}$$

Theorem: Let $\bar{\Phi}$ and Φ be the $(k+1)$–dimensional moving and fixed surfaces with the leading curves $\bar{y}$, y such that $\bar{y}$ and y are the moving and the fixed pole curves of $\bar{\Phi}$ and Φ, respectively. If $\{\bar{e}_1, \dots, \bar{e}_k\}$ and $\{e_1, \dots, e_k\}$ are the principal frames of $\bar{\Phi}$ and Φ then we have the following results [1]

$$\begin{aligned} &\text{i.} && \bar{\alpha}_{\sigma\nu} = \alpha_{\sigma\nu},\ [\leq \sigma \leq m,\ 1 < \nu < k \\ &\text{ii.} && \bar{\alpha}_{(m+\rho)}\cdot = \alpha_{(m+\rho)}\,\sigma,\ 1 < \rho < k-m \\ &\text{iii.} && \bar{K}_\sigma = K_\sigma \\ &\text{iv.} && A\bar{a}_{k+\sigma} = a_{k+\sigma},\ 1 < \sigma < m. \end{aligned} \tag{12}$$

Let a 2–ruled surface (not cylinder) φ in E^n be given. Then the magnitude $b = \xi / K$ is called the Blaschke invariant of φ where ξ and K are obtained from the following equations for $k = 1$ [2]:

$$\dot{e}_\sigma = \sum_{\nu=1}^{k} \alpha_{\sigma\nu} e_\nu + K_\sigma a_{k+\sigma},\ K_\sigma > 0,\ 1 \leq \sigma \leq m\ . \tag{13}$$

$$\dot{y} = \sum_{\nu=1}^{k} \xi_\nu e_\nu + \eta_{m+1} a_{k+m+1},\ \eta_{m+1} \neq 0.$$

Let Φ be a $(k+1)$–ruled surface in E^n. The dimension of the asymptotic bundle of Φ being $k+m$, $m > 0$ the magnitudes

$$b_i = \xi_i / K_i,\ 1 \leq i \leq m \tag{14}$$

are called the principal Blaschke invariants of Φ and also

$$B = \sqrt[m]{|b_1 \dots b_m|} \tag{15}$$

is called the Blaschke invariant of Φ [5].

In the case $m = k$, the central ruled surface $\Omega \subset \varnothing$ degene rates in the line of striction. Thus, the Blaschke invariant b of the 2–ruled surface ψ generated by 1–dimensional supspace $E(t) = Sp\{e(t)\} \subset E_k(t)$ can be given by

$$b = \frac{\sum_{\nu=1}^{k} \xi_\nu \cos\theta_\nu}{\sqrt{\sum_{\mu=1}^{k}\left[\left(\sum_{\nu=1}^{k} \cos\theta_\nu \alpha_{\nu\mu}\right)^2 + (\cos\theta_\mu K_\mu)^2\right]}}$$

where $e(t) = \sum_{\nu=1}^{k} \cos\theta_\nu e_\nu$, θ_ν = constant,, $\| e \| = 1$ [5].

2. On The Blaschke Invariants Of The Pair Of Generalized Ruled Surfaces Under The Homothetic Motion In The Euclidean n-Space E^n

In this section we discus the relations between the Blaschke invariants of the fixed ruled surfaces and of the moving ruled surfaces under the homothetic motion in E^n.

Let $\bar{\Phi}$ and Φ be the $(k+1)$–dimensional moving and fixed ruled surfaces with the leading curves $\bar{y}$ and y, where $\bar{y}$ and y are the moving and the fixed pole curves under the homothetic motion, respectively. And suppose that dimension of the asymptotic bundle of $\bar{\Phi}$ is $k + m$, $m > 0$. Then it can be easily shown that the dimension of the asymptotic bundle of Φ is also $k + m$. Let $\{\bar{e}_1, \ldots, \bar{e}_k\}$ and $\{e_1, \ldots, e_k\}$ be the ONF of the generators $\bar{E}_k(t) \subset \bar{\Phi}$ and $E_k(t) \subset \Phi$, respectively. For the leading curves we have, from (1), (5) and (6)

$$\dot{y} = S\bar{y},\ S = h\dot{A}. \tag{17}$$

On the other hand from (8), (13) and (17) we obtain that

$$\xi_\nu = h\ \bar{\xi}_\nu. \tag{18}$$

If $\bar{b}_i$ and b_i are the i^{th} principal Blaschke invariants of the moving and fixed ruled surfaces $\bar{\Phi}$ and Φ, respectively, from (12) and (18) we get

$$b_i = h\bar{b}_i \tag{19}$$

so we have the following results:

Corollary 1: We have the following relations between the principal Blaschke invariants of the moving $(k + 1)$–ruled surface $\bar{\Phi}$ and of the fixed $(k + 1)$–ruled surface Φ under the homothetic motion

$$b_i = h\bar{b}_i.$$

Corollary 2: $\bar{B}$ and B being the Blaschke invariants of $\bar{\Phi}$ and Φ, respectively, we have

$$B = h\bar{B}.$$

Let $e_k(t)$ be a unit vector in the generator $E_k(t)$ satisfying

$$e = \sum_{\nu=1}^{k} \cos \theta_\nu e_\nu, \quad \theta_\nu = \text{constant}. \tag{20}$$

Under the homothetic motion then we obtain a 2–ruled surface generated by $E(t) = Sp\ \{e(t)\}$ with the leading curve y in the fixed space E, say ψ. If we put $S(\bar{e}) = he$, in the same way, we obtain a 2–ruled surface $\bar{\psi}$ generated by $\bar{E}(t) = Sp\ \{\bar{e}\}$ with the leading curve $\bar{y}$ in the moving space $\bar{E}$ where $\bar{y}$ and y are the moving and the fixed pole curves, respectively. We have

$$\cos \bar{\theta}_\nu = \cos \theta_\nu, \; 1 \leq \nu \leq k. \tag{21}$$

Using (12), (21) and (18) in (16) we obtain

$$b = \frac{h \sum_{\nu=1}^{k} \bar{\xi}_\nu \cos\theta_\nu}{\sqrt{\sum_{\mu=1}^{k} \left[\left(\sum_{\nu=1}^{k} \cos\bar{\theta}_\nu \bar{\alpha}_{\nu\mu} \right)^2 + (\cos\bar{\theta}_\mu \bar{K}_\mu)^2 \right]}} \tag{22}$$

or

$$b = h\bar{b}. \tag{23}$$

Corollary 3: The relation (23) holds between the Blaschke invariants of $\bar{\psi}$ and ψ.

In this case, taking $\bar{e} = \bar{e}_i$, $1 \leq i \leq m$ we obtain (19) from (22). So (22) can be considered as a generalization of (19).

HOMOTETİK HAREKETLER ALTINDA GENELLEŞTİRİLMİŞ REGLE YÜZEYLERİN BLASCHKE İNVARYANTLARI ÜZERİNE

ÖZET

Bu çalışmada, homotetik hareketler altındaki hareketli ve sabit $\bar{\Phi}$ ve Φ $(k + 1)$–regle yüzey çiftlerinin Blaschke invaryantları arasında

bazı bağıntılar bulundu. Ayrıca $\overline{\psi} \subset \overline{\Phi}$ ve $\psi \subset \Phi$ 2–regle yüzey çiftlerinin Blaschke invaryantları arasındaki ilişkiler elde edildi. Bu bulunan ilişkilerden, özel bir hal olarak asli Blaschke invaryantları arasındaki bağıntılar elde edildi.

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