

ON THE 3-PLANE AND CONE AND SPHERES

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SUMMARY

Elkholy and Areefi showed that in a space time, the intersection of a plane, passing through the origin, with the light cone, given by the equation $\sum_{i=1}^3 x_i^2 - x_4^2 = 0$, is two 2-planes perpendicular to each other. In this study, instead of Elkholy-Areefi's light cone in a space time by dealing with the cone given by the equation, $a'_1 x_1^2 + a'_1 x_2^2 + a'_1 x_3^2 - b x_4^2 = 0$ and showing that also its intersection with 3-plane, passing through the origin, is two 2-planes perpendicular to one another, the generalization of the article of Elkholy-Areefi has been obtained. Furthermore, validity is proved for the sphere given by the equation

$$\sum_{i=1}^3 x_i^2 + \alpha = x_4^2.$$

I. Introduction

1.2. Definition

A diametral plane is known by the equation

$$l \frac{\partial F}{\partial x} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} \quad (1)$$

where

$$F(x,y,z)=ax+by+cz+2fyz+2gzx+2hxy+d=0 \quad [1].$$

Calculating $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$, the equation of di-

ametral plane is obtained as

$$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)=0. \quad (2)$$

1.2. Definition :

If the normal of a diametral plane is linearly dependent to the vector (l, m, n) , then the diametral plane given by (2) is called perpendicular to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} .$$

If the diametral plane is perpendicular to the line

$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ then the homogeneous system of linear equations,

$$\left. \begin{aligned} (a-\lambda) l + hm + gn &= 0 \\ hl + (b-\lambda) m + fn &= 0 \\ gl + fm + (c-\lambda)n &= 0 \end{aligned} \right\} \quad (3)$$

is obtained. To have non-trivial solutions, the coefficient determinant must be zero for this equation system. That is,

$$\lambda^3 - \lambda^2(a+b+c) + \lambda(bc+ca+ab-h^2-g^2-f^2) - D = 0 \quad (4)$$

where,

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} .$$

1.3. Definition

The equation,

$$\lambda^3 - \lambda^2(a+b+c) + \lambda(bc+ca+ab-h^2-g^2-f^2) - D = 0$$

is called the cubic discriminating of $F(x, y, z)$ [1].

Regarding to the equation in (4), for λ there are at most three solutions. For each λ_i , $1 \leq i \leq 3$, we can find the three non-trivial solutions (l_i, m_i, n_i) . So,

$$l_i x + m_i y + n_i z = 0, \quad 1 \leq i \leq 3,$$

diametral planes are obtained.

II. The Main Results

Let

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0$$

be a 3-plane passing through the origin. Getting $\frac{A_i}{A_4} = B_i$,

$1 \leq i \leq 3$, the equation of this plane becomes,

$$\sum_{i=1}^3 B_i x_i + x_4 = 0. \quad (5)$$

On the other hand, for the cone given by

$$\sum_{i=1}^3 a_i' x_i^2 - b x_4^2 = 0$$

substituting $\frac{a_i'}{b} = a_i$, the equation reduces to,

$$\sum_{i=1}^3 a_i x_i^2 - x_4^2 = 0. \quad (6)$$

From (5) and (6) we have that

$$\left(\sum_{i=1}^3 B_i^2 x_i \right)^2 = \sum_{i=1}^3 a_i x_i^2$$

or

$$\Rightarrow \sum_{i=1}^3 (B_i^2 - a_i) x_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^3 B_i B_j x_i x_j = 0,$$

If we denote $B_i^2 - a_i = C_i^2$, we have the quadric

$$\begin{aligned} F(x_1, x_2, x_3) = & C_1^2 x_1^2 + C_2^2 x_2^2 + C_3^2 x_3^2 + 2B_1 B_2 x_1 x_2 + \\ & 2B_1 B_3 x_1 x_3 + 2B_2 B_3 x_2 x_3 = 0. \end{aligned} \quad (7)$$

The diametral plane of this quadric can be given as,

$$l \frac{\partial F}{\partial x_1} + m \frac{\partial F}{\partial x_2} + n \frac{\partial F}{\partial x_3} = 0. \quad (8)$$

Calculating $\frac{\partial F}{\partial x_i}$, $1 \leq i \leq 3$, we can have

$$\begin{aligned} (C_1^2 l + B_1 B_2 m + B_1 B_3 n) x_1 + (B_1 B_2 l + C_2^2 m + B_2 B_3 n) x_2 + \\ (B_1 B_3 l + B_2 B_3 m + C_3^2 n) = 0. \end{aligned} \quad (9)$$

If we consider that this diametral plane is perpendicular to the line

$$\frac{x_1}{l} = \frac{x_2}{m} = \frac{x_3}{n}$$

then we have,

$$\begin{aligned} \frac{C_1^2 l + B_1 B_2 m + B_1 B_3 n}{l} &= \frac{B_1 B_2 l + C_2^2 m + B_2 B_3 n}{m} \\ &= \frac{B_1 B_3 l + B_2 B_3 m + C_3^2 n}{n} = \lambda, \end{aligned}$$

and therefore we can write the homogeneous system of linear equations,

$$\left. \begin{aligned} (C_1^2 - \lambda)l + B_1 B_2 m + B_1 B_3 n &= 0 \\ B_1 B_2 l + (C_2^2 - \lambda)m + B_2 B_3 n &= 0 \\ B_1 B_3 l + B_2 B_3 m + (C_3^2 - \lambda)n &= 0 \end{aligned} \right\}. \quad (10)$$

The cubic discriminating of the equation (7) is

$$\begin{aligned} \lambda^3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) + \lambda(C_2^2 C_3^2 + C_1^2 C_2^2 + C_1^2 C_3^2 - \\ B_2^2 B_3^2 - B_2^2 B_3^2 - B_1^2 B_2^2) - D = 0 \end{aligned} \quad (11)$$

where

$$D = \begin{vmatrix} C_1^2 & B_1 B_2 & B_1 B_3 \\ B_1 B_2 & C_2^2 & B_2 B_3 \\ B_1 B_3 & B_2 B_3 & C_3^2 \end{vmatrix}$$

Substituting $B_i^2 = C_i^2 + a_i$, equation (11) becomes,

$$\begin{aligned} \lambda^3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) - \lambda[(a_2 + a_3)C_1^2 + (a_1 + a_3)C_2^2 + a_1 a_2 + a_1 a_3 + \\ a_2 a_3 + (a_1 + a_2)C_3^2] - [a_2 a_3 C_1^2 + a_1 a_3 C_2^2 + a_1 a_2 C_3^2 + 2a_1 a_2 a_3] = 0. \end{aligned} \quad (12)$$

As a special case if we take $a_1 = a_2 = a_3 = a$ in (6) the equation (12) becomes

$$\begin{aligned} \lambda^3 - \lambda^2(C_1^2 + C_2^2 + C_3^2) - \lambda[2a(C_1^2 + C_2^2 + C_3^2) + 3a^2] - [a^2(C_1^2 + C_2^2 \\ + C_3^2) + 2a^3] = 0. \end{aligned}$$

For the sake of shortness, if we denote, $A=C_1^2+C_2^2+C_3^2$, then the cubic equation becomes,

$$\lambda^3 - \lambda^2 A - \lambda [2a \cdot A + 3a^2] - [a^2 A + 2a^3] = 0 \quad (13)$$

or

$$(\lambda + a)^2 [\lambda - (A + 2a)] = 0 .$$

It follows that

$$\lambda_1 = \lambda_2 = -a , \text{ and } \lambda_3 = A + 2a .$$

Using $\lambda_1 = \lambda_2 = -a$ in equation (10) we have

$$B_1 l_1 + B_2 m_1 + B_3 n_1 = 0 . \quad (14)$$

And using $\lambda_3 = A + 2a$ in (10) we have,

$$\left. \begin{aligned} -(B_2^2 + B_3^2) l_3 + B_1 B_2 m_3 + B_1 B_3 n_3 &= 0 \\ B_1 B_2 l_3 - (B_1^2 + B_3^2) m_3 + B_2 B_3 n_3 &= 0 \\ B_1 B_3 l_3 + B_2 B_3 m_3 - (B_1^2 + B_2^2) n_3 &= 0 \end{aligned} \right\} . \quad (15)$$

Dividing the first equation of (15) by B_1 and the second by B_2 then subtracting, we have

$$\frac{l_3}{B_1} = \frac{m_3}{B_2} . \quad (16)$$

Again from the second and third equations, we can have

$$\frac{m_3}{B_2} = \frac{n_3}{B_3} \quad (17)$$

and then from (16) and (17)

$$\frac{l_3}{B_1} = \frac{m_3}{B_2} = \frac{n_3}{B_3} = k, \quad k \in \mathbb{R} . \quad (18)$$

On the other hand, from solution (10) we have the diametral planes

as

$$\left. \begin{aligned} l_1 x_1 + m_1 x_2 + n_1 x_3 &= 0 \\ l_3 x_1 + m_3 x_2 + n_3 x_3 &= 0 \end{aligned} \right\} \quad (19)$$

and from (18), these equations reduces to

$$\left. \begin{aligned} l_1 x_1 + m_1 x_2 + n_1 x_3 &= 0 \\ B_1 x_1 + B_2 x_2 + B_3 x_3 &= 0 \end{aligned} \right\}$$

where

$$\langle (l_1, m_1, n_1), (B_1, B_2, B_3) \rangle = l_1 B_1 + m_1 B_2 + n_1 B_3 \quad (20)$$

and from (14) it vanishes. So the planes given by (19) are perpendicular to each other.

As a result, we can write the following theorem:

II.1. Theorem :

The intersection of the cone $\sum_{i=1}^3 a_i' x_i^2 = b x_4^2$ and the 3-plane

$\sum_{i=1}^4 A_i x_i = 0$ is two 2-planes, perpendicular to each other, if $a_1' = a_2' = a_3'$.

By using the S_α sphere $\sum_{i=1}^3 \alpha_i^2 + \alpha = \alpha_4^2$ instead of the cone (6) all of the results are valid. So we can express the following theorem:

II.2. Theorem :

In $(3+1)$ -spacetime, the intersection of the sphere $\sum_{i=1}^3 x_i^2 + \alpha = x_4^2$ and the hyperplane $\sum_{i=1}^4 A_i x_i = 0$ is two 2-planes, which are perpendicular to each other.

REFERENCES

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