

ON THE GAUSSIAN AND MEAN CURVATURES OF PARALLEL HYPERSURFACES

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ABSTRACT

In this study, we give a generalization of the higher order Gaussian curvature function and the mean curvature function about the parallel surfaces in E^3 , for E^{n+1} .

BASIC CONCEPTS

DEFINITION 1: Let M_1 and M_2 are two hypersurfaces in E^{n+1} , with unit normal vector N_1 of M_1

$$N_1 = \sum_{i=1}^{n+1} a_i \frac{\partial}{\partial x_i} .$$

Where each a_i is a C^∞ function on M_1 . If there is a function f , from M_1 to M_2 such that

$$\begin{aligned} f : M_1 &\longrightarrow M_2 \\ P &\longrightarrow f(P) = (p_1 + r a_1(P), \dots, p_{n+1} + r a_{n+1}(P)). \end{aligned}$$

Then M_2 is called a parallel hypersurfaces of M_1 , where $r \in \mathbb{R}$ [1].

DEFINITION 2: Let M be a hypersurface in E^{n+1} and S denotes the shape operator on M , at $P \in M$. The function H defined by

$$\begin{aligned} H : M &\longrightarrow \mathbb{R} \\ P &\longrightarrow H(P) = \text{Tr } S(P) \end{aligned}$$

is called the mean curvature function of M and the real number $H(P)$ is called mean curvature of M at the point P [3].

DEFINITION 3: Let M be a hypersurface in E^{n+1} and S denotes the shape operator on M , at $P \in M$. The function K defined by

$$\begin{aligned} K : M &\longrightarrow R \\ P &\longrightarrow K(P) = \det S(P) \end{aligned}$$

is called the Gaussian curvature function of M and the $K(P)$ is called Gaussian curvature of M at the point P [3].

DEFINITION 4: Let M be a hypersurface in E^{n+1} and $T_M(P)$ be a tangent space on M , at $P \in M$. If S_P denotes the shape operator on M , at $P \in M$, then

$$S_P: T_M(P) \longrightarrow T_M(P)$$

is a linear mapping. If we denote the characteristic values by $\lambda_1, \lambda_2, \dots, \lambda_n$ and the corresponding characteristic vectors by x_1, x_2, \dots, x_n of S_P then $\lambda_1, \lambda_2, \dots, \lambda_n$ are the principal curvatures and x_1, x_2, \dots, x_n are the principal directions of M , at $P \in M$. On the other hand, if we use the notations

$$K_1(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i$$

$$K_2(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i < j} \lambda_i \lambda_j$$

$$K_3(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i < j < t} \lambda_i \lambda_j \lambda_t$$

$$K_n(\lambda_1, \lambda_2, \dots, \lambda_n) = \prod_{i=1}^n \lambda_i$$

then the characteristic polynomial of $S(P)$ becomes

$P_{S(P)}(\lambda) = \lambda^n + (-1) K_1 \lambda^{n-1} + \dots + (-1)^n K_n$ and K_i , $1 \leq i \leq n$ are uniquely determined, where the functions K_i are called the higher ordered Gaussian curvatures of the hypersurface M [4], [5].

THEOREM 1: Let f be a mapping from M to M_r and M_r be a parallel hypersurface of M . Then f preserves the principal curvature directions [2]

THEOREM 2: Let M_r be a parallel surface of the surface $M \subset E^3$. Let the Gaussian curvature and mean curvature of M be denoted by K and H at $P \in M$, respectively and the Gaussian curvature and mean curvature of M_r be denoted by K_r and H_r at $f(P) \in M_r$, respectively. Then we know that

$$K_r = \frac{K}{1 + r H + r^2 K}$$

and

$$H_r = \frac{H + 2 r K}{1 + r H + r^2 K}$$

[2].

GENERALIZED THEOREMS

THEOREM 1: Let M be a hypersurface of E^{n+1} and K_1, K_2, \dots, K_n are the higher order Gaussian curvatures and k_1, k_2, \dots, k_n the principal curvatures at the point $P \in M$.

Let us define a function

$$\Phi : M \longrightarrow R$$

such that $P \longrightarrow \Phi(P) = \Phi(r, k_1, k_2, \dots, k_n)$

$$= \prod_{i=1}^n (1 + r k_i)$$

$$\begin{aligned} \Phi(r, k_1, k_2, \dots, k_n) &= 1 + r \sum_{i=1}^n k_i + r^2 \sum_{i < j} k_i k_j + \dots \\ &\quad + r^n \prod_{i=1}^n k_i \end{aligned}$$

or

$$\Phi(r, k_1, k_2, \dots, k_n) = 1 + r K_1 + r^2 K_2 + \dots + r^n K_n,$$

where $r \in R$ is given in Definition 1.

PROOF: We prove the theorem by induction method.

For $n = 1$, the theorem holds.

Actually,

$$\begin{aligned} \Phi(r, k_1, k_2, \dots, k_n) &= \prod_{i=1}^1 (1 + r k_i) \\ &= 1 + r k_1 \\ &= 1 + r \sum_{i=1}^1 k_i \\ &= 1 + r K_1. \end{aligned}$$

now Suppose that the theorem holds for $n=p$ and show that it is true for $n=p+1$:

$$\begin{aligned}
 \Phi(r, k_1, k_2, \dots, k_n) &= \prod_{i=1}^p (1 + rk_i) \\
 &= 1 + r \sum_{i=1}^p k_i + r^2 \sum_{i<j}^p k_i k_j + \dots \\
 &\quad + r^p \prod_{i=1}^p k_i \\
 &= 1 + r K_1 + r^2 K_2 + \dots + r^p K_p.
 \end{aligned}$$

For $n = p + 1$, we have

$$\begin{aligned}
 \Phi(r, k_1, k_2, \dots, k_n) &= \prod_{i=1}^{p+1} (1 + rk_i) \\
 &= \left[\prod_{i=1}^p (1 + rk_i) \right] (1 + rk_{p+1}) \\
 &= \left(1 + r \sum_{i=1}^p k_i + r^2 \sum_{i<j}^p k_i k_j + \dots \right. \\
 &\quad \left. + r^p \prod_{i=1}^p k_i \right) (1 + rk_{p+1}) \\
 &= 1 + r \left(\sum_{i=1}^p k_i + k_{p+1} \right) + r^2 \left(\sum_{i<j}^p k_i k_j \right. \\
 &\quad \left. + k_{p+1} \sum_{i=1}^p k_i \right) \\
 &\quad + \dots + r^p \left(\sum_{i<j<\dots<s}^p k_i k_j \dots k_s \right. \\
 &\quad \quad \quad \text{p times} \\
 &\quad \left. + k_{p+1} \sum_{i<j<\dots<l}^p k_i k_j \dots k_l \right) \\
 &\quad \quad \quad \text{p-1 times}
 \end{aligned}$$

$$\begin{aligned}
 & + r^{p+1} k_{p+1} \prod_{i=1}^p k_i \\
 & = 1 + r \sum_{i=1}^{p+1} k_i + r^2 \sum_{i<j}^{p+1} k_i k_j \\
 & + \dots + r^p \sum_{i<j<\dots<s}^{p+1} k_i k_j \dots k_s + r^{p+1} \prod_{i=1}^{p+1} k_i \\
 & \qquad \qquad \qquad \text{p times} \\
 & = 1 + r K_1 + r^2 K_2 + \dots + r^p K_p + r^{p+1} K_{p+1},
 \end{aligned}$$

which proves the theorem.

THEOREM 2: Let M_r be a hypersurface to the parallel hypersurface M of E^{n+1} . K_1, K_2, \dots, K_n denote the higher order Gaussian curvatures of M , at $P \in M$. K_r and H_r generalized Gaussian and mean curvature of M_r , at the point $f(P)$.

$$\begin{aligned}
 \text{Suppose the function } \Phi : M & \longrightarrow R \\
 P & \longrightarrow \Phi(P) = \Phi(r, k_1, k_2, \dots, k_n) \\
 & = \prod_{i=1}^n (1 + rk_i).
 \end{aligned}$$

Then we have

$$K_r = \frac{\partial^n \Phi(r, k_1, k_2, \dots, k_n)}{(\partial r)^n (n!) \Phi(r, k_1, k_2, \dots, k_n)}$$

and

$$H_r = \frac{\partial \Phi(r, k_1, k_2, \dots, k_n)}{\partial r \Phi(r, k_1, k_2, \dots, k_n)}.$$

PROOF: Let X_1, X_2, \dots, X_n are the principal directions and k_1, k_2, \dots, k_n principal curvatures of M , at $P \in M$.

Let $f_*(X_1), f_*(X_2), \dots, f_*(X_n)$ be principal directions at the point $f(P)$ of M_r then we know that [2].

$$S_r(f_* X_1) = \frac{k_1}{1 + rk_1} f_* X_1$$

$$S_r (f_* X_2) = \frac{k_2}{1+rk_2} f_* X_2$$

⋮

$$S_r (f_* X_n) = \frac{k_n}{1+rk_n} f_* X_n.$$

Then, we know that, the shape operator of M_r is

$$S_r = \begin{bmatrix} \frac{k_1}{1+rk_1} & 0 & \dots & 0 \\ 0 & \frac{k_2}{1+rk_2} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \frac{k_n}{1+rk_n} \end{bmatrix}$$

and

$$\begin{aligned} K_r &= \det S_r \\ &= \frac{k_1}{1+rk_1} \cdot \frac{k_2}{1+rk_2} \cdots \frac{k_n}{1+rk_n} \\ &= \frac{k_1 \cdot k_2 \cdots k_n}{\prod_{i=1}^n (1+rk_i)} \\ &= \frac{\prod_{i=1}^n k_i}{\prod_{i=1}^n (1+rk_i)} \end{aligned}$$

and

$$\Phi (r, k_1, k_2, \dots, k_n) = \prod_{i=1}^n (1+rk_i)$$

then

$$K_r = \frac{\frac{\partial^n \Phi (r, k_1, k_2, \dots, k_n)}{(\partial r)^n}}{(n!) \Phi (r, k_1, k_2, \dots, k_n)} .$$

On the other hand

$$\begin{aligned} H_r &= I_z S_r \\ &= \frac{k_1}{1+rk_1} + \frac{k_2}{1+rk_2} + \dots + \frac{k_n}{1+rk_n} \\ &= \frac{k_1 \prod_{i=2}^n (1+rk_i) + (1+rk_1)k_2 \prod_{i=3}^n (1+rk_i) + \dots + k_n \prod_{i=1}^{n-1} (1+rk_i)}{\prod_{i=1}^n (1+rk_i)} . \end{aligned}$$

Since

$$\Phi (r, k_1, k_2, \dots, k_n) = \prod_{i=1}^n (1+rk_i) ,$$

we get that

$$H_r = \frac{\frac{\partial \Phi (r, k_1, k_2, \dots, k_n)}{\partial r}}{\Phi (r, k_1, k_2, \dots, k_n)} .$$

The case $n = 2$ of this theorem reduces to the result which is given in [2].

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