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PROOF OF BIEBERBACH'S CONJECTURE: A SURVEY

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INTRODUCTION

In a recent paper (c. f., Uluçay (1985)), which has been abstracted in the Book of ABSTRACTS of the ICM 86, we have shown the redundancy of the hypothesis that **n–1 is prime** in the Lemma XXXI of Shaeffer and Spencer (1950). It should be also noted the evident fact that when $P_2(z)$ is reduced to $1 + \lambda z + \mu z^2$, the possibility of a double root has been tacitly overlooked by Shaeffer and Spencer (1950). Yet, in view of Lemma XXXI, the conclusive Thecrem II of Uluçay (1973) just corresponds to that exceptional case which reduces the extremal function $\sigma(z)$ to koebe function. Below is reproduced the ABSTRACT in question.

Let $w = f(z) \in S$ statisfy two δ_n–equations, one which is of degree n and the other of degree k, $2 \leqq k \leqq n-1$. Then, it is shown that the hypothesis that **n–1 is prime** introduced by Shaeffer and Spencer (1950) in their lemma is redundant, and therefore the rationality of f (z) does not follow from the latter hypothesis. The authors claim that the equations

$$\beta_1^{n-1} = \beta_1^{k-1} = 1 \tag{1}$$

give $\beta_1 = 1$, because $k < n$ and n–1 is prime.

Yet, the following examples show the redundancy of the hypothesis: (i) **k=n–1.** In theis case, (1) implies $\beta_1^{n-1} = \beta_1^{n-2} = 1$. Hence $1=\beta_1^{n-1} = \beta_1\beta_1^{n-2} = \beta_1$. (ii) **k = 2.** In this case (1) implies $\beta_1^{n-1} = \beta_1=1$. (iii) The **Koebe function** is rational and satisfies every δ_n–equation of degree n with $A_{n-1} = B_{n-1} = 1$, n–1 not necessarily prime. **General case.** Finally it is shown that each root of $\beta_1^{n-1} = 1$ is a root of $\beta_1^{k-1} = 1$. This is however impossible, unless $\beta_1 = 1$, since the roots of unity are all distinct and $k < n$.

Hence in all cases the hypothesis n–1 is prime is redundant.

The conclusive Theorem II of Uluçay (1973) corresponds to k=n–1 (not necessarily prime).

Omitting all the details having no direct contribution to the proof, the following survey is an exact reproduction of the original proof which seems to be the most natural and shortest possible proof of Bieberbach's Conjecture (c.f., Uluçay (1973)). We shall use the notations of our original proof.

The proof (Theorem II) will follow immediately as soon as we prove.

THEOREM I: Let $\sigma(z) = z + \sigma_2 z^2 + \ldots + \dot{\sigma}_n z^n + \ldots \varepsilon S$, $\dot{\sigma}_n = \sup |a_n|$, $\dot{\sigma}_n \geqq n$, be an extremal function. Then, $(\sigma_2, \ldots, \sigma_{n-1})$ is a boundary point of the coefficient region V_{n-1}.

PROOF:

1. **The 2-dimensional cross-sections** π^*. Suppose on the contrary that $(\sigma_2, \ldots, \sigma_{n-1})$ is an interior point of V_{n-1}. Then there exists a bounded schlicht function $f(z) = z + \sigma_2 z^2 + \ldots + \sigma_{n-1} z^{n-1} + b_n z^n + \ldots$ belonging to the point $(\sigma_2, \ldots, \sigma_{n-1}) \in V_{n-1}$. Conversely, since f (z) is bounded, it belongs also to the interior point $(\sigma_2, \ldots, \sigma_{n-1}, b_n) \in V_n$ [Uluçay (1973) pp.4, 9, 10].

The proof will now rest upon a careful approach to the boundary point $p \in V_n$ belonging to the extremal function via the 2 - dimensional cross-sections π^*.

Let then π denote the 2–dimensional cross-section obtained by holding $\sigma_2, \ldots, \sigma_{n-1}$ fixed and varying the last coordinate. π passes through $p = (\sigma_2, \ldots, \sigma_{n-1}, \dot{\sigma}_n)$ εbd V_n which satisfies a differential equation δ_n of the form [Shaeffer and Spencer (1950) pp. 36–44 and Uluçay (1973) p.4]

$$((z/w)(dw/dz))^2 p(w) = Q(z), \quad |z| < 1, \; w = \sigma(z)$$

$$p(w) = \sum_{\nu=1}^{n-1} A_\nu / w^\nu, \quad Q(z) = \sum_{\nu=-(n-1)}^{n-1} B_\nu / z^\nu.$$

Here Q (z) is analytic with at least one multiple zero z_0 on $|z| = 1$ which must be of even order, say m.

It is known that π is convex (Uluçay (1973), p. 3).

Next, for $\varepsilon \geqq 0$ sufficiently small, $e^{-i\varepsilon}f(e^{i\varepsilon} z)$ being also bounded and schlicht the neighboring 2-dimensional cross-section π^*, obtained by holding $\sigma_2^*, \ldots, \sigma^*_{n-1}$ fixed, passes through the neighboring point $p^* = (\sigma^*_2, \ldots, \sigma^*_{n-1}, \sigma^*_n)$ where (Uluçay (1973), p. 11),

$$\sigma^*_\nu = \sigma_\nu e^{i(\nu-1)\varepsilon},\ \nu = 2, \ldots, n-1,\ \sigma^*_n = \dot{\sigma}_n e^{i(n-1)\varepsilon}$$

Now, p^*, belonging to the extremal function $\sigma^*(z) = e^{-i\varepsilon}\sigma(e^{i\varepsilon}z)$ is necessarily also a boundary point of V_n. Hence, in the neighborhood of p, the set (π^*) yields for all $\varepsilon \geqq 0$ sufficiently small a set (γ^*) of convex arcs passing through (p^*) respectively.

2. **The differentiable closed set of points $\bar{R}$Cbd V_n.** We deduce the following important consequence that the set (γ^*) sweeps on bd V_n near p a differentiable closed set of points $\bar{R}$Cbd V_n containing p (Uluçay (1973), pp. 13–15). For, the generating point $a^* = a^*(\varepsilon, \alpha) \in \bar{R}$ has partial derivatives with respect to ε, α.

In fact we have $a^* = (\sigma_2^*, \ldots, \sigma^*_{n-1}, a_n^*)$ with $a^*_n = a^*_n(\varepsilon\ \alpha)$ $= \alpha^*_n + i\beta^*_n$, $\alpha^*_n = \dot{\sigma}_n - \lambda^*_n$,

$$\lambda^*_n = (\beta_n + \dot{\sigma}_n \sin(n-1)\varepsilon)\cos(\alpha - (n-1)\varepsilon) / \sin\alpha,$$
$$\beta^*_n = (\beta_n + \dot{\sigma}_n \sin(n-1)\varepsilon)\sin(\alpha-(n-1)\varepsilon) / \sin\alpha. \tag{2}$$

The derivation of formulas (2) is very simple. We only have to recall that as the point $a = (\sigma_2, \ldots, \sigma_{n-1}, a_n)$, $|a_n| \leqq \dot{\sigma}_n$ describes γ, its projection a_n will describe in the complex plane the arc Γ lying in the disc G centre at the origin , radius $\dot{\sigma}_n$, and passing through $\dot{\sigma}_n$ (Uluçay (1973), p. 8). Clearly, as a consequence of the convexity, Γ cannot lie on the real axis (Uluçay (1973), p. 6, (2)). Near $\dot{\sigma}_n$, Γ has no point in common with the circumference g of G neither (Uluçay (1973), p. 8). Nevertheless, again as a consequence of the convexity (Uluçay (1973), p. 6, (2)), Γ must be tangent to g at $\dot{\sigma}_n$, i.e., is differentiable at $\dot{\sigma}_n$ as expected. Let then Γ_1 be that part of Γ that lies, say in the lower part of the disc G, with one end point at $\dot{\sigma}_n$. Near $\dot{\sigma}_n$, we define $a_n \in \Gamma_1$ as follows:

Denote by $\tau = \bar{\xi}\dot{\sigma}_n$, $\xi = e^{i(n-1)\varepsilon}$, $\varepsilon > 0$ sufficiently small, the point on the circle g of G. Let δ be a straight line from τ in G making the angle $0 < \alpha < \pi / 2$ with the straight line issuing from τ and parallel to the real axis. Then a_n lies at the intersection of δ and Γ_1. Accordingly, upon the rotation ξ, (10) $a^*_n = \xi a_n$ and (11) $\beta^*_n = \lambda^*_n \tan(\alpha-(n-1)\varepsilon)$ yield (2). Here a^*_n is at the intersection of $\Gamma_1^* = \xi\Gamma_1$ issuing from $\xi\dot{\sigma}_n = \sigma^*_n$ and $\delta^* = \xi\delta$ issuing from $\dot{\sigma}_n$ and which makes the angle $\alpha - (n-1)\varepsilon$ with the real axis.

Note that the derivation of formulas (2) is crucial. For, the differentiability of γ^* alone is not sufficient to ensure the differentiability of $\bar{R}$.

3. **The differential equation δ^*_n and $Q^*(z)$.** Now, each point $a^* \in \bar{R}$ satisfies a neighboring differential equation δ^*_n of the same form as δ_n (Schaeffer and Spencer (1950), pp. 36–43 and Uluçay (1973) pp. 12–13) with

(A) $A^*_v = \Sigma^n_{k=v+1}\ \sigma^*_k{}^{(v+1)}\ F^*_k$

(B) $B^*_v = \sum_{k=1}^{n-v} k\sigma^*_k F^*_{k+v}\ ,\ B^*_o = \sum_{k=2}^{n} (k-1)\,\sigma^*_k F^*_k\ ,\ B^*_o > 0$

4. **The relation $z^{n-1}\ Q^*(z) = R_*$.** At each point a^*, $Q^*(z)$ is analytic with at least one multiple zero on $|z| = 1$ which must be of even order, and tending to z_0 with $Q^*(z)$ tending to $Q(z)$ uniformly as as $\varepsilon \to 0$.

We finally write the important obvious relation that will ultimately solve the Bieberbach's Conjecture, i.e., (Uluçay (1973), pp. 15–17))

$$z^{n-1}Q^*(z) = \overset{*}{B}_{n-1}z^{2n-2} + \ldots + \overset{*}{B}_o z^{n-1} + \ldots \overset{*}{B}_{n-1}.B^*_o > 0$$

Thus the polynomial R_* on the right has the same zeros on $|z| = 1$ as $Q^*(z) = 0$.

At a multiple zero, the discriminant D of the polynomial R_* vanishes, and $D = 0$ is an homogeneous polynomial of degree 4n–6, i.e., an algebraic variety (Uluçay (1973), p. 17).

Now, $\bar{R}$ being differentiable, the vector F^* is uniquely determined at each point a^* (Uluçay (1973), p. 14, p. 17 and Shaeffer and Spencer (1950), p. 111).

5. **$\bar{R}$ is homeomorphic to the algebraic variety $\bar{N} C E^{2n-1}$.** It then follows from (B) that $\bar{R}$ is homeomorphic to a closed set $\bar{N}$ of vectors $B^* = (B^*_o, B^*_1, \ldots B^*_{n-1}) \in E^{2n-1}$ containing B and on which D vanishes. Hence $\bar{N}$ is an algebraic variety (Uluçay (1973), p. 17) and therefore a compact analytic variety (see, in particular Chow (1949), p. 893 and

Uluçay (1973), p. 17). Therefore on $\overline{N}$ there is an analytic arc σ with one end point at B along which the coordinates $B^*\upsilon$, $\nu = 0, 1, \ldots, n-1$, can be expressed analytically with respect to some parameter (Uluçay (1973) p. 17).

6. **The Contradiction via the Fundamental Theorem.** But, in view of a fundamental theorem (Lindelöf (1947), p. 26), in a sufficiently small neighborhood of z_0, $z^{n-1} Q^*(z) = R_*$ has along σ m distinct roots which are analytic functions of the parameter and tending to z_0 as $B^* \to B$.

Hence $Q^*(z)$ has along σ near z_0 on $|z| = 1$ zeros of order at most 1. This contradiction proves Theorem I.

THEOREM II: Let $p = (\sigma_2, \ldots, \sigma_{n-1}, \dot{\sigma}_n)$. Then p belongs to the Koebe function with $\dot{\sigma}_n = n$.

PROOF: Theorem I implies that the extremal function $\sigma(z)$ satisfies two differential equations δ_n, δ_{n-1} (Uluçay (1973), p.18; (16), (15)) respectively. It is found that $\sigma(z)$ is of the form $z / (1-e^{i\beta} z)(1-e^{i\alpha} z)$ which maps $|z| < 1$ onto a domain whose entire boundary lies on a straight line through the origin. But $\sigma(z)$ being extremal this boundary lies on a single radial line (Uluçay (1973), p. 5, pp. 18–19 and Schaeffer and Spencer (1950), pp. 144–158).

Thus $\alpha = \beta$, and $\sigma(z)$ is the Koebe function.

REMARK: It is apparent that the proof of Theorem I contains a shorter one involving directly Γ_1. For., the vanishing of D on the topological image $\mathcal{K}_1$ of Γ_1 turns $\mathcal{K}_1$ into an algebraic and therefore an analytic arc, thereby yielding the same contradiction (Uluçay (1973), pp. 16–17).

EN RESUME: This survey gives to the author the occasion to express his indebtness to the ICM–86 for the invitation and the acceptance of the abstract of the paper entitled "On A Lemma of Shaeffer and Spencer"

The Abstract with its conclusive Theorem II is significant in three respects. It announces via the ICM–86

(i) that the first rigorous proof of Bieberbach's Conjecture has been already published at Ankara, TURKEY in 1973.

(ii) that the proof is the most natural and shortest possible. In fact, it can be read at once from the conclusive Theorem II: The extremal function σ (z) is the Koebe function. In fact the proof follows readily from the three characteristic properties of σ (z) (max $|a_n|$):

a) The boundary of σ (z) consists of a single analytic slit extending to infinity and without any critical point (see Proof of Uluçay (1973), pp. 4–5)

b) σ (z) satisfies a δ_n–equation of degree n (loc. cit. p. 18, formula (16)).

c) $\sigma(z)$ satisfies further a δ_n-equation of degree n–1 (loc. cit. p. 18, formula (15), Theorem I).

(iii) Theorem II discloses for the first time an error in Lemma XXXI of Shaeffer and Spencer undiscovered by the experts since 1950, i.e., that the hypothesis 'n–1 is a prime' is redundant. And so the rationality of f and in particular $f(z) = z/1 + \lambda z + \mu z^2 = z/ (1-e^{i\alpha} z)(1-e^{i\beta} z)$ does not follow from the hypothesis as claimed by some expert even to day.

The possible case $\alpha = \beta$ was tacitly overlooked by Shaeffer and Spencer (1950) and which shows that the lemma is self contradictory.

Theorem II is in fact an existence theorem corresponding to the case $k = n - 1$ (not necessarily prime).

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