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ON MAPPINGS WHOSE POWERS ARE CONTRACTIONS ON A METRIC SPACE

by

T. SOM and R.N. MUKHERJEE

Faculté des Sciences de l'Université d'Ankara
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T. SOM and R.N. MUKHERJEE

Applied Mathematics Section, School of Applied Sciences Institute of Technology, Banaras Hindu University Varanasi - 221 005, *INDIA*

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ABSTRACT

In the present paper we give results to show that a fixed power of a mapping satisfying generalized contraction type of condition of Pal and Maiti [4] or Das [1] or Jaggi [2] is a contraction of Banach type under some given conditions. In another section we generalize further the result of Sastry and Naidu [6] condisering two mappings on a metric space and get a result where a fixed power of a composite map is a contraction under a given condition. The result is based on the idea of generalized orbit (to be introduced later) of two mappings.

1. INTRODUCTION

After the mid half of the last decade the Banach contraction theorem has been generalized in different ways by many authors and as a result we have many generalized contractive mappings on a metric space. Recently Rao [5] gave a result, which reduces the nth (fixed) power of a generalized Kannan type mapping to be the Banach contraction under a condition given by him. We further go ahead in this direction and show that the nth (fixed) power of a mapping satisfying generalized contraction type of condition of Pal and Maiti [4] and Jaggi [2] becomes a contraction of Banach type under a given condition.

Theorem 1. Let T be a mapping on a metric space (X,d) into itself satisfying any one of the following three inequalities

(i) $d(x,Tx)+d(y,Ty) \leq \beta \ \{d(x,Ty)+d(y,Tx)+d(x,y)\}, \ \frac{1}{2} \leq \beta < \frac{2}{3}$

(ii) $d(x,Tx)+d(y,Ty)+d(Tx,Ty) \leq \gamma \ \{ d(x,Ty)+d(yTx), 1 \leq \gamma < \frac{3}{2}$

(iii) $d(Tx,Ty) \leq \alpha \ \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta \ d(x,y), \ x \neq y, \ 0 \leq \alpha+\beta < 1$

with $d(x, p) < d(x, Tx) + d(Tx, p)$ or,

$$d(Tx, p) < d(Tx, x) + d(x, p) \text{ for all } x \neq p \ (=Tp) \in X.$$

Further if there exists $h > 0$ such that

$$d(x,Tx)+d(y,Ty) \leq h\ d(x,y),\ x\neq y \tag{1}$$

then T^n is a contraction for a large n in all the above three cases.

Proof. Let x_0 be any arbitrary point in X, we define a sequence $\{x_n\}$ as follows.

$$x_1 = Tx_0,\ x_2 = Tx_1,\ \ldots,\ x_n = Tx_{n-1} = T^n x_0,\ \ldots$$

Then from [4] and [2] it is easy to see that $\{x_n\}$ is a Cauchy sequence in all the above three cases. Now assuming X to be complete we get a point t in X such that $x_n \to t$ in each case. Further for any positive integer p we have

$$d(x_n,x_{n+p}) \leq d(x,x_{n+1})+(x_{n+1},x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p})$$

$$\leq \frac{\lambda^n}{1-\lambda}\ d(x_0,Tx_0)$$

where $\lambda = \frac{2\beta-1}{1-\beta}$, $\frac{\gamma-1}{2-\gamma}$ and $\frac{\beta}{1-\alpha}$ in case (i), (ii) and (iii)

respectively. Now as p tends to infinity we get,

$$d(T^n x_0,t) \leq \frac{\lambda^n}{1-\lambda}\ d(x_0,Tx_0) \tag{2}$$

Similarly for any arbitrary y_0, we can get

$$d(T^n y_0,t) \leq \frac{\lambda^n}{1-\lambda}\ d(y_0,Ty_0) \tag{3}$$

Adding (2) and (3) we get

$$d(T^n x_0,T^n y_0) \leq \frac{\lambda^n}{1-\lambda}\ \{d(x_0 Tx_0)+d(y_0,Ty_0)\}$$

$$\leq \frac{h\lambda^n}{1-\lambda}\ d(x_0,y_0)$$

or, $d(T^n x_0,T^n y_0) \leq M_n\ d(x_0\ y_0)$

It easily follows that $M_n < 1$ for some large n and hence T^n is a contraction in each case.

Next suppose that X is not complete. Then in case (i)

$$\begin{aligned} d(Tx\,Ty) &\leq d(Tx,x)+d(x,y)+d(y,Ty) \\ &\leq \beta\,\{d(x,Ty)+d(y,Tx)+d(x,y)\} + d(x,y) \\ &\leq \{\beta\,(h+3)+1\}\; d(x,y) \end{aligned}$$

since $d(x,Ty)+d(y,Tx) \leq (h+2)d(x,y)$ from (1). Next in case (ii) we have

$$\begin{aligned} 2d(Tx,Ty) &= d(Tx,Ty)+d(Tx,Ty) \\ &\leq d(Tx,x)+d(x,y)+d(y,Ty)+d(Tx,Ty) \\ &\leq \gamma\;\{d(x,Ty)+d(y,Tx)\}+d(x,y) \\ &\leq \{\gamma(h+2)+1\}\; d(x,y) \end{aligned}$$

And in case (iii) we get

$$\begin{aligned} d(Tx,Ty) &\leq \frac{\alpha\; d(x,Tx)d(y,Ty)}{d(x,y)} + \beta\; d(x,y) \\ &\leq \frac{\alpha\;[\{d(x,Tx)+d(y,Ty)\}^2 - \{d(x,Tx)-d(y,Ty)\}^2]}{4} \cdot \frac{1}{d(x,y)} + \beta\; d(x,y) \\ &\leq \frac{\alpha\;\{h\; d(x,y)\}^2}{4\; d(x,y)} + \beta\; d(x,y) \\ &\leq \left(\frac{\alpha\; h^2}{4} + \beta\right)\; d(x,y) \end{aligned}$$

We see that in the above three cases T is uniformly continuous. Let $\tilde{X}$ and $\tilde{T}$ be the completions of X and T respectively. Then clearly $\tilde{T}$ will satisfy the inequalities (including (1)) considered in the theorem and therefore it follows from what is proved above for T that $\tilde{T}^n$ is a contraction for some large n. Hence T^n is a contraction for a large n.

In our next theorem we search for another condition under which $(T^m)^n$, where T^m satisfying inequality (A) of Theorem 1 of Das [1] is a contraction.

Theorem 2. Let T be a self mapping on a metric space (X,d). Let T^m (denoting it by S), for some positive integer m, satisfies

$$d(Sx,Sy) \leq \alpha_1 \frac{d(x,Sx)d(y,Sy)}{d(x,y)} + \alpha_2 \frac{d(x,Sx)d(y,Sx)}{d(Sx,Sy)}$$

$$+ \alpha_3 \frac{d(x,Sy)d(y,Sy)}{d(Sx,Sy)} + \beta_1 d(x,y) +$$

$$\beta_2 d(x,Sx) + \beta_3 d(ySy) + \beta_4 d(x,Sy) + \beta_5 d(y,Sx)$$

for all $x,y \in X$ with $x \neq y$, $Sx \neq Sy$ where $\sum_{i=1}^{3} \alpha_i + \sum_{j=1}^{5} \beta_j < 1$

$\alpha_i, \beta_j > 0$, $i = 1,2,3$ and $j = 1,2,\ldots,5$.

If there exist $h > 0$, $k > 0$ such that

$$d(x,Sx) + d(y,Sy) \leq hd(x,y),\ x \neq y \tag{4}$$

and

$$\frac{d(x,Sx)d(y,Sx)}{d(Sx,Sy)} + \frac{d(x,Sy)d(y,Sy)}{d(Sx,Sy)} \leq k\, d(x,y)\, d(Sx,Sy) \tag{5}$$

Then S^n is a contraction for some large n.

Proof: Without any loss of generality we take $\alpha_2 = \alpha_3, \beta_2 = \beta_3, \beta_4 = \beta_5$. Assuming X to be complete we get S^n is a contraction for some large n by arguments analogous to that used in the proof of Theorem 1. Now when X is not complete, we have

$$d(Sx,Sy) \leq \frac{\alpha_1\, d(x,Sx)d(y,Sy)}{d(x,y)} + \frac{\alpha_2\, d(x,Sx)d(y,Sx)}{d(Sx,Sy)}$$

$$+ \frac{\alpha_2\, d(x,Sy)d(y,Sy)}{d(Sx,Sy)} + \beta_1 d(x,y) + \beta_2 d(x,Sx)$$

$$+ \beta_2 d(y,Sy) + \beta_4 d(x,Sy) + \beta_4 d(y,Sx)$$

$$\leq \frac{\alpha_1 \{h\, d(x,y)\}^2}{4d(x,y)} + \frac{\alpha_2\, k\, d(x,y)d(Sx,Sy)}{d(Sx,Sy)}$$

$$+ \beta_1 d(x,y) + \beta_2 h\, d(x,y) + \beta_4 (h+2)d(x,y)$$

$$\leq \left(\frac{\alpha_1 h^2}{4} + k\,\alpha_2 + \beta_1 + \beta_2 h + (h+2)\,\beta_4\right) d(x,y)$$

$$\leq K\,d(x,y), \text{ where } 0 < K = \left(\frac{\alpha_1 h^2}{4} + \alpha_2 k + \beta_1 + \beta_2 h + \beta_4 (h+2)\right)$$

Thus S is uniformly continuous. Then the similar arguments as given in the proof of Theorem 1 lead that S^n is a contraction for some large n.

2. In what follows we give a generalization of Theorem 1 of Sastry and Naidu [6]. In the generalized contraction of a single mapping of [6] has been extended further for two mappings involving two different composite structures and then we show that a fixed powers of these composite maps respectively again are contractions under a given condition. The concept of generalized orbit of two mappings, which is defined below, is used in the theorem.

Definition: Let f, g be two self mappings of a complete metric space (X,d). Let F = g f be the composite map of f and g. Then the generalized orbit of a point $x \in X$ is defined to be the sequence of iterates $\{x, f(x), g f(x) = F(x), f F(x), F^2(x), f F^2(x), \ldots\}$ to be denoted by $D_g(x)$.

Theorem 3. Let f, g : X $\to$ X, where (X,d) is a metric space and f, g commute with each other. Let $\delta(A)$ denotes the diameter of a non-empty subset A of X and for any x, y in X

$$\beta(x,y) = \inf_{1 \leq n < \infty} \{d(x,F^n x),\ d(x,F^n y),\ d(x,f\,F^{n-1} x),$$
$$d(x,f\,F^{n-1} y),\ d(y,F^n x),\ d(y,f\,F^{n-1} x),\ d(y,F^n y),\ d(y,f\,F^{n-1} y)\}$$

Further we suppose that

$$\delta(D_g(x)) < \infty \tag{6}$$

and

$$d(F x, F y) \leq \alpha\, \delta\,(D_g(x) \cup D_g(y)),\ 0 \leq \alpha < 1 \tag{7}$$

$$d(f F x, f F y) \leq \beta\, \delta(D_g(x) \cup D_g(y)),\ 0 \leq \beta < 1 \tag{8}$$

and there exists h > 0 such that

$$\beta(x,y) \leq h\, d(x,y),\ x \neq y \tag{9}$$

Then F^n and $f\,F^{n-1}$ ($\equiv$ G) are contractions for some large n.

Proof: Let A be a generalized invariant subset of X under f and g, then (7) and (8) implies that

$$\delta(F(A)) \leq \alpha\, \delta(A) \tag{10}$$

and

$$\delta(f\, F(A)) \leq \beta\, \delta(A) \tag{11}$$

Further for $x,y \in X$, let $B = D_g(x) \cup D_g(y)$ such that B is F and $f\,F$ invariant. Then from (10) and (11) we get

$$\delta(F^n(B)) \leq \alpha^n \delta(B) \ \forall\ n \geq 1 \tag{12}$$

and

$$\delta(f\, F^{n-1}(B)) \leq \beta\, \alpha^{n-2}\, \delta(B) \ \forall\ n > 1 \tag{13}$$

where

$$\begin{aligned}\delta(B) = \sup_{1\leq n<\infty} \{&d(x,F^n x), d(x,F^n y), d(x,f\,F^{n-1} x),\\ &d(x,f\,F^{n-1} y), d(y,F^n x), d(y,F^n y),\\ &d(y,f\,F^{n-1} x), d(y,f\,F^{n-1} y)\}\end{aligned} \tag{14}$$

Also $\delta(B) < \infty$ by (6). Then for $n \geq 1$ using (12) and (13) we get

$$d(x,F^n(x)) \leq d(x,y) + K(m) + \alpha\, \delta(B) \ \forall\ m \geq 1 \tag{15}$$

where $K(m)$ is any one of $d(x,F^m x)$, $d(x,F^m y)$,

$d(x,f\,F^{m-1} x), d(x,f\,F^{m-1} y), d(y\, F^m x)$, $d(y,F^m y), d(y,f\,F^{m-1} x)$,

and

$d(y,f\,F^{m-1} y)$. Take $K(m) = d(y,f\,F^{m-1} x)$

for one verification. Then due to $f\,g = g\,f$ we have

$$\begin{aligned}\text{and } d(x,F^n x) &\leq d(x,y) + d(y,f\,F^{m-1} x) + d(f\,F^{m-1} x,F^n x)\\ &\leq d(x,y) + d(y,f\,F^{m-1} x) + d(F^{m-1}(fx),F^n x)\\ &\leq d(x,y) + d(y,f\,F^{m-1} x) + \alpha\, \delta(B).\end{aligned}$$

Thus

$$d(x,F^n x) \leq d(x,y) + \beta(x,y) + \alpha\, \delta(B) \tag{16}$$

Further we observe that if the left hand side of (15) is replaced by any one of $d(x,F^n y)$, $d(x,f\,F^{n-1} x)$, $d(x,f\,F^{n-1} y)$, $d(y,F^n x)$, $d(y,F^n y)$, $d(y,f\,F^{n-1} x)$, $d(y,f\,F^{n-1} y)$ it remains true. Hence from (14) we get

$$\delta(B) \leq d(x,y) + \beta(x,y) + \alpha\, \delta(B)$$

or,

$$\delta(B) \leq \frac{1}{(1-\alpha)} [d(x,y) + \beta(x,y)]$$

Then (12) and (13) further implies that

$$\delta(F^n(B)) \leq \frac{\alpha^n}{(1-\alpha)} [d(x,y) + \beta(x,y)] \leq \frac{\alpha^n(1+h)}{(1-\alpha)} d(x,y)$$

and

$$\delta(f F^{n-1}(B)) \leq \frac{\beta\alpha^{n-2}}{(1-\alpha)} [d(x,y) + \beta(x,y)] \leq \frac{\beta\alpha^{n-2}(1+h)}{(1-\alpha)} d(x,y)$$

for $x \neq y$ from (9). Therefore we have

$$d(F^n x, F^n y) \leq L\, d(x,y) \;\forall\; x,y \in X$$

and

$$d(f,F^{n-1}x, f F^{n-1}y) \leq M\, d(x,y) \;\forall\; x,y \in X$$

where $L = \frac{\alpha^n(1+h)}{(1-\alpha)}$ and $M = \frac{\beta\alpha^{n-2}(1+h)}{(1-\alpha)}$ are less than 1 for some large n and hence F^n and $f F^{n-1}$ are contractions for some large n and this completes the whole proof of the theorem.

Example: Let $X = \{1,2,3,4\}$, $d(1,2) = 4$, $d(1,3) = 1.5$ $d(1,4) = 2.6$, $d(2,3) = 2.5$, $d(2,4) = 1.4$, $d(3,4) = 3$.

Define $T: X \to X$ by $T(1) = 1$, $T(2) = 3$, $T(3) = T(4) = 1$.

Then T satisfies condition (i) of Theorem 1 for $\frac{8}{13} \leq \beta < \frac{2}{3}$

Clearly T is not a contraction (for the pair (2,4)) on X but we observe that T^2 is a contraction.

REFERENCES

1. DAS, P., Some results on fixed points in complete metric spaces, Indian J. Pure Appl. Math., 13 (3) (1982), 303–308.

2. JAGGI, D.S., Some unique fixed point theorems, Indian J. Pure Appl. Math., 8(2) (1977), 223–230.

3. KANNAN, R., Some results on fixed points, Bull. Cal. Math. Soc., 60 (1968), 71–76.

4. PAL, T.K., and MAITI, M., Extensions of fixed point theorems of Rhoades and Ciric, Proc. Amer. Math. Soc. 64 (2), (1977), 283–286.

5. RAO, J.M., Mappings on metric spaces, Indian J. Pure Appl. Math., 11(4) (1980), 502–506.

6. SASTRY, K.P.R., and NAIDU, S.V.R., On contraction maps, Indian J. Pure Appl. Math., 13 (12) (1982), 1417–1419.