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ON THE EXPLICIT ANALYTICAL SOLUTION OF THE (TRUNCATED HYPEREXPONENTIAL QUEUES)

By

Al Seedy, R.O. and Abou-El-Ata, M.O.

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ON THE EXPLICIT ANALYTICAL SOLUTION OF THE TRUNCATED HYPEREXPONENTIAL QUEUES

Al Seedy, R.O.* and Abou–El–Ata, M.O.**

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ABSTRACT

The aim of this paper is to find P_{nj} the probabilities that there are n units in the system of the truncated hyperexponential queue in an explicit form for each branch j. Also the corresponding measures of effectiveness. In fact this problem had been treated earlier either in an implicit form as in Gupta or numerically as in White and el.

INTRODUCTION

Many researchers treated the problem of solving the truncated hyperexponential queues. Nishida [3] analized the queue: $H_2/M/1/N$ in an implicit form. This works has been followed by Gupta [1] who discussed the general queue: $H_k/M/1/N$ and gave both the probability generating function and the expected number in the system in an explicit form but not the probabilities. And to find the probabilities he said that it may be evaluated numerically when the maximum queue allowed is finite. And so White and el [4] had followed his advice and solved some cases for N, C = 1, 2 numerically using a computer program and matrices Also Gupta and Goyal [2] solved the dual queue: $M/H_k/1/N$ analytically in an implicit form. In this paper we treat two different queues analytically to find P_{nj} the probability of n units in the system and the next arrival unit occupies branch j in some cases using the elementary methods of matrices. In fact we treat both the queues: $H_2/M/C/N$ for C, N = 1, 2 and $H_2/E_2/1/1$. Also we find the corresponding measures of effectiveness of each queue in an explicit form.

* Dept. of Math., Faculty of Edu., Menofia Univ. Egypt.

** Dept. of Math., Faculty of Science, Zagazig Univ. Egypt.

2. The Truncated Queue: $H_2/M/C/N$

Consider the truncated C–channels hyperexponential interarrival queue having two branches with arrival rates: $\alpha\lambda_1$, $(1-\alpha)\lambda_2$ and the service time is an exponential with rate μ. The discipline considered is FIFo. Let us define:

p_{nj} = the steady–state probability of finding the system in state (n,j),

where n is the number of units in the system and j the arrival branch is occupied by the next arrival unit.

As in White and el [4] we can write the steady–state difference equations of the probabilities for the queue: $H_2/M/C/N$ as follows:

$$\left.\begin{aligned} \lambda_1 p_{01} &= \mu\, p_{11} \\ \lambda_2 p_{02} &= \mu\, p_{12} \end{aligned}\right\} \quad n = 0 \qquad (1)$$

$$\left.\begin{aligned} (\lambda_1 + n\mu)p_{n1} &= \alpha\lambda_1 p_{n-1,1} + \alpha\lambda_2 p_{n-1,2} + (n+1)\mu p_{n+1,1} \\ (\lambda_2 + n\mu)p_{n2} &= (1-\alpha)\lambda_1 p_{n-1,1} + (1-\alpha)\lambda_2 p_{n-1,2} + (n+1)\mu p_{n+1,2} \end{aligned}\right\} \quad 1 \leq n < c \qquad (2)$$

$$\left.\begin{aligned} (\lambda_1 + C\mu)p_{n1} &= \alpha\lambda_1 p_{n-1,1} + \alpha\lambda_2 p_{n-1,2} + C\mu p_{n+1,1} \\ (\lambda_2 + C\mu)p_{n2} &= (1-\alpha)\lambda_1 p_{n-1,1} + (1-\alpha)\lambda_2 p_{n-1,2} + C\mu p_{n+1,2} \end{aligned}\right\} \quad C \leq n \leq N\text{-}1 \qquad (3)$$

$$\left.\begin{aligned} (\lambda_1 + C\mu)p_{N1} &= \alpha\lambda_1 p_{N-1,1} + \alpha\lambda_2 p_{N-1,2} + \alpha\lambda_1 p_{N1} + \alpha\lambda_2 p_{N2} \\ (\lambda_2 + C\mu)p_{N2} &= (1-\alpha)\lambda_1 p_{N-1,1} + (1-\alpha)\lambda_2 p_{N-1,2} + (1-\alpha)\lambda_1 p_{N1} + (1-\alpha)\lambda_2 p_{N2} \end{aligned}\right\} \quad n = N \qquad (4)$$

These are $2(N+1)$ equations in the same number of probabilities. Really one of them is redundant and we use the boundary condition $\sum_n \sum_j p_{nj} = 1$ instead of it. In fact we prove the following powerful lemma that helps us to solve this system in a more easier way.

Lemma: For the truncated C–channels hyperexponential queue: $H_2/M/C/N$ we have:

$$\sum_{n=0}^{N} p_{n1} = \frac{\alpha\rho_2}{\alpha\rho_2 + (1-\alpha)\rho_1} = 1 - \sum_{n=0}^{N} p_{n2} \qquad (5)$$

where $\rho_i = \dfrac{\lambda_i}{\mu}$, $i = 1, 2$.

Proof. Adding the first relation in the above four relations (1) → (4) we get:

$$(1-\alpha)\lambda_1 \sum_{n=o}^{N} p_{n1} = \alpha\lambda_2 \sum_{n=o}^{N} p_{n2}$$

and since $\displaystyle\sum_{n=o}^{N} (p_{n1} + p_{n2}) = 1$, then

$$\sum_{n=o}^{N} p_{n1} = \frac{\alpha\lambda_2}{\alpha\lambda_2 + (1-\alpha)\lambda_1} \text{ , and ;}$$

$$\sum_{n=o}^{N} p_{n2} = \frac{(1-\alpha)\lambda_1}{\alpha\lambda_2 + (1-\alpha)\lambda_1}$$

Dividing the numerator and the denominator by μ we get relation (5). In a similar manner we can obtain the same result if we add the second relation in relations (1) → (4).

Corollary: Let $\lambda_1 = 2\alpha\lambda$, $\lambda_2 = 2\ (1-\alpha)\lambda$ in the above lemma we get:

$$\sum_{n=o}^{N} p_{n1} \cdot = \sum_{n=o}^{N} p_{n2} = \tfrac{1}{2} \qquad (6)$$

Some Special Cases

1. Case The truncated queue: $H_2/M/1/1$

Let $C = N = 1$ in equations (1) → (4) we get:

$$-\rho_1 p_{o1} + p_{11} = 0 \qquad (7)$$

$$-\rho_2 p_{o2} + p_{12} = 0 \quad , \rho_i = \frac{\lambda_i}{\mu}, i = 1, 2. \qquad (8)$$

$$-(\rho_1 + 1)p_{11} + \alpha\rho_1 p_{o1} + \alpha\rho_2 p_{o2} + \alpha\rho_1 p_{11} + \alpha\rho_2 p_{12} = 0 \qquad (9)$$

$$-(\rho_2+1)p_{12}+(1-\alpha)\rho_1 p_{o1}+(1-\alpha)\rho_2 p_{o2}+(1-\alpha)\rho p_{11}+(1-\alpha)\rho_2 p_{12} = 0 \quad (10)$$

To solve these four equations we replace the last two equations by those of the lemma for $N = 1$.

i.e

$$p_{o1} + p_{11} = \frac{\alpha\rho_2}{\alpha\rho_2 + (1-\alpha)\rho_1} \tag{11}$$

$$p_{o2} + p_{12} = \frac{(1-\alpha)\rho_1}{\alpha\rho_2 + (1-\alpha)\rho_1} \tag{12}$$

Solving (7), (11) and (8), (12) we get:

$$p_{o1} = \frac{\alpha\rho_2}{x(1+\rho_1)}, \quad p_{11} = \frac{\alpha\rho_1\rho_2}{x(1+\rho_1)}, \text{ and;} \tag{13}$$

$$p_{o2} = \frac{(1-\alpha)\rho_1}{x(1+\rho_2)}, \quad p_{12} = \frac{(1-\alpha)\rho_1\rho_2}{x(1+\rho_2)}. \tag{14}$$

from which we can find the delay probability and the expected number in the system and the queue as:

$$p_o = p_{o1} + p_{o2} = \frac{x + \alpha\rho_2^2 + (1-\alpha)\rho_1^2}{x(1+\rho_1)(1+\rho_2)} \tag{15}$$

$$L = \sum_{n=1}^{1} n(p_{n1} + p_{n2}) = \frac{\rho_1 \cdot \rho_2 \cdot (1+x)}{x(1+\rho_1)(1+\rho_2)}, \text{ and; } L_q = 0 \tag{16}$$

where: $x = \alpha\rho_2 + (1-\alpha)\rho_1$.

Corollary : If $\alpha = 1$ (or $\alpha = \frac{1}{2}$ or $\alpha = 0$) we can get $p_o = \frac{1}{1+\rho}$,

$L = \frac{\rho}{1+\rho}$, $\rho = \frac{\lambda}{\mu}$ the measures of effectiveness of the queue: M/M/1/1.

2. Case The truncated queue: $H_2/M/1/2$

Let $C = 1$ and $N = 2$ in equations (1) $\rightarrow$ (4) we get:

$$-\rho_1 p_{o1} + p_{11} = 0 \tag{17}$$

$$-\rho_2 p_{o2} + p_{12} = 0 \tag{18}$$

$$-(\rho_1 + 1)p_{11} + \alpha\rho_1 p_{o1} + \alpha\rho_2 p_{o2} + p_{21} = 0 \tag{19}$$

$$-(\rho_2+1)p_{12}+(1-\alpha)\rho_1 p_{01}+(1-\alpha)\rho_2 p_{02}+p_{22}=0 \tag{20}$$

$$-(\rho_1+1)p_{21}+\alpha\rho_1 p_{11}+\alpha\rho_2 p_{12}+\alpha\rho_1 p_{21}+\alpha\rho_2 p_{22}=0 \tag{21}$$

$$-(\rho_2+1)p_{22}+(1-\alpha)\rho_1 p_{11}+(1-\alpha)\rho_2 p_{12}+(1-\alpha)\rho_1 p_{21}+(1-\alpha)\rho_2 p_{22}=0 \tag{22}$$

To solve these six equations we replace the last two equations by those of the lemma for $N = 2$.

i.e.

$$p_{01}+p_{11}+p_{21}=\frac{\alpha\rho_2}{x} \tag{23}$$

$$p_{02}+p_{12}+p_{22}=\frac{(1-\alpha)\rho_1}{x} \tag{24}$$

To find p_{n1} the probabilities of the first branch we solve equations (17), (19) and (23) by the matrices technique as follows:

$$\begin{array}{ccc} p_{01} & p_{11} & p_{21} \end{array}$$
$$\left[\begin{array}{ccc|c} \rho_1 & -1 & 0 & 0 \\ \alpha\rho_1 & -(\rho_1+1) & 1 & -\alpha\rho_2 p_{02} \\ 1 & 1 & 1 & \frac{\alpha\rho_2}{x} \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} H & 0 & 0 & T \\ 1-\alpha\rho_1 & \rho_1+2 & 0 & E \\ 1 & 1 & 1 & K \end{array}\right]$$

where

$$H=\frac{(1+\rho_1)^2-\alpha\rho_1}{\rho_1+2}, \quad T=\frac{\alpha\rho_2(1+xp_{02})}{x(\rho_1+2)},$$

$$E=\alpha\rho_2\left(p_{02}+\frac{1}{x}\right), \text{ and; } K=\frac{\alpha\rho_2}{x}.$$

Then we get

$$p_{01}=\frac{T}{H}=\frac{\alpha\rho_2(1+xp_{02})}{x\,[(1+\rho_1)^2-\alpha\rho_1]} \tag{25}$$

$$p_{11}=\frac{\alpha\rho_1\rho_2}{(1+\rho_1)^2-\alpha\rho_1}\left[1+\frac{p_{02}}{x}\right] \tag{26}$$

$$p_{21} = \frac{\alpha\rho_2}{(1+\rho_1)^2 - \alpha\rho_1} \cdot \left[\frac{\rho_1(\rho_1 - \alpha + 1)}{x} - (1+\rho_1)p_{o2}\right] \quad (27)$$

To find p_{n2} the probabilities of the second branch, we solve (18), (20) and (24) using (25) and the matrices technique as follows:

$$\begin{array}{ccc} p_{o2} & p_{12} & p_{22} \end{array}$$

$$\left[\begin{array}{ccc|c} \rho_2 & -1 & 0 & 0 \\ -a & (\rho_2+1) & -1 & b \\ 1 & 1 & 1 & C \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} [(\rho_2+1)^2-a] & 0 & 0 & b+c \\ 1-a & \rho_2+2 & 0 & b+c \\ 1 & 1 & .. & C \end{array}\right]$$

where

$$a = \frac{(1-\alpha)\,\rho_2\,(1+\rho_1)^2}{(1+\rho_1)^2 - \alpha\rho_1}, \quad b = \frac{\alpha(1-\alpha)\,\rho_1\rho_2}{x\,([1+\rho_1)^2-\alpha\rho_1]}, \text{ and;}$$

$$C = \frac{(1-\alpha)\rho_1}{x}.$$

Then we find:

$$p_{o2} = \frac{(1-\alpha)\rho_1}{y}\,[(1+\rho_1)^2 + \alpha(\rho_2-\rho_1)], \quad (28)$$

$$p_{12} = \frac{(1-\alpha)\,\rho_1\,\rho_2}{y}\,[(1+\rho_1)^2 + \alpha(\rho_2-\rho_1)], \text{ and;} \quad (29)$$

$$p_{22} = \frac{(1-\alpha)\rho_1}{y}\,[(1+\rho_2)^2\{(1+\rho_1)^2-\alpha\rho_1\} - (1-\alpha)\rho_2(1+\rho_1)^2 - (1+\rho_2)\{(1+\rho_1)^2+\alpha(\rho_2-\rho_1)\}] \quad (30)$$

where

$$y = x\,[(1+\rho_2)^2\,\{(1+\rho_1)^2-\alpha\rho_1\} - (1-\alpha)\rho_2(1+\rho_1)^2]$$

Thus we can find more explicit formulas for p_{o1}, p_{11} and p_{21} using relation (28).

The delay probability is:

$$p_o = p_{o1} + p_{o2} = \frac{1}{y}\,[\alpha\rho_2(1+\rho_2)^2 + (1-\alpha)\,\{\rho_1\,(1+\rho_1)^2 - \alpha(\rho_2-\rho_1)^2] \quad (31)$$

Also the expected number in both the system and the queue are:

$$L = \sum_{n=1}^{2} n(p_{n1} + p_{n2})$$

$$= \frac{1}{y} \left[2y+\rho_1\rho_2\{\alpha(1+\rho_2)^2+(1-\alpha)(1+\rho_1)^2\} -2\alpha\rho_2(1+\rho_1)\{(1+\rho_2)^2 +(1-\alpha)(\rho_1-\rho_2)\} -2(1-\alpha)\rho_1(1+\rho_2)\{(1+\rho_1)^2+\alpha(\rho_2-\rho_1)\}\right] \quad (32)$$

$$L_q = \sum_{n=1}^{2} (n-1)(p_{n1} + p_{n2})$$

$$= \frac{1}{y} \left[y-\alpha\rho_2(1+\rho_1)\{(1+\rho_2)^2+(1-\alpha)(\rho_1-\rho_2)\}-(1-\alpha)\rho_1(1+\rho_2)\{(1+\rho_1)^2+\alpha(\rho_2-\rho_1)\}\right] \quad (33)$$

Corollary: Let $\alpha = 1$ (or $\alpha = \frac{1}{2}$ or $\alpha = 0$) we get:

$$p_o = \frac{1}{1+\rho+\rho^2}, \quad L = \frac{\rho(1+2\rho)}{1+\rho+\rho^2}, \text{ and; } L_q = \frac{\rho^2}{1+\rho+\rho^2}$$

which are the measures of effectiveness of the queue: M/M/1/2.

3. Case The truncated queue: $H_2/M/2/2$

Let C = N = 2 in equations (1) → (4) we get:

$$-\rho_1 p_{o1} + p_{11} = 0 \quad (34)$$

$$-\rho_2 p_{o2} + p_{12} = 0 \qquad \rho_i = \frac{\lambda_i}{\mu}, \; i = 1,2 \quad (35)$$

$$-(\rho_1+1)p_{11} + \alpha\rho_1 p_{o1} + \alpha\rho_2 p_{o2} + 2p_{21} = 0 \quad (36)$$

$$-(\rho_2+1)p_{12} + (1-\alpha)\rho_1 p_{o1} + (1-\alpha)\rho_2 p_{o2} + 2p_{22} = 0 \quad (37)$$

$$-(\rho_1+2)p_{21} + \alpha\rho_1 p_{11} + \alpha\rho_2 p_{12} + \alpha\rho_1 p_{21} + \alpha\rho_2 p_{22} = 0 \quad (38)$$

$$-(\rho_2+2)p_{22}+(1-\alpha)\rho_1 p_{11}+(1-\alpha)\rho_2 p_{12}+(1-\alpha)\rho_1 p_{12}+(1-\alpha)\rho_2 p_{22} = 0 \quad (39)$$

As before we replace the last two equations by those of the lemma for N = 2.

i.e.

$$p_{o1} + p_{11} + p_{21} = \frac{\alpha\rho_2}{x} \quad (40)$$

$$p_{o2} + p_{12} + p_{22} = \frac{(1-\alpha)\,\rho_1}{x} \tag{41}$$

To find p_{n1} the probabilities of the first branch we solve (34), (36) and (40) using the matrices methods as follows:

$$\begin{array}{ccc} p_{o1} & p_{11} & p_{21} \end{array}$$

$$\left[\begin{array}{ccc|c} \rho_1 & -1 & 0 & 0 \\ \alpha\rho_1 & -(\rho_1+1) & 2 & -\alpha\rho_2 p_{o2} \\ 1 & 1 & 1 & F \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} Z & 0 & 0 & \alpha\rho_2 p_{o2}+2F \\ (2-\alpha\rho_1) & (\rho_1+3) & 0 & \alpha\rho_2 p_{o2}+2F \\ 1 & 1 & 1 & F \end{array}\right]$$

where

$$F = \frac{\alpha\rho_2}{x}, \text{ and; } Z = (\rho_1+1)\,(\rho_1+2)-\alpha\rho_1.$$

From the above matrix we get:

$$p_{o1} = \frac{\alpha\rho_2}{(1+\rho_1)\,(2+\rho_1)-\alpha\rho_1}\left(\frac{2}{x} + p_{o2}\right) \tag{42}$$

$$p_{11} = \frac{\alpha\rho_1\rho_2}{(1+\rho_1)\,(2+\rho_1)-\alpha\rho_1}\left(\frac{2}{x} + p_{o2}\right) \tag{43}$$

$$p_{21} = \frac{\alpha\rho_2}{(1+\rho_1)\,(2+\rho_1)-\alpha\rho_1}\left[\rho_1\,(1+\rho_1-\alpha)-(1+\rho_1)p_{o2}\right] \tag{44}$$

Also to find p_{n2} the probabilities of the second branch we solve (35), (37) and (41) using (42) and matrix technique. Then we have:

$$\begin{array}{ccc} p_{o2} & p_{12} & p_{22} \end{array}$$

$$\left[\begin{array}{ccc|c} \rho_2 & -1 & 0 & 0 \\ a & -(\rho_2+1) & 2 & -b \\ 1 & 1 & 1 & C \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} [(\rho_2+1)\,(\rho_2+2)-a] & 0 & 0 & b+2C \\ (2-a) & (\rho_2+3) & 0 & b+2C \\ 1 & 1 & 1 & C \end{array}\right]$$

where

$$a = \frac{(1-\alpha)\rho_2(\rho_1+1)\,(\rho_1+2)}{(\rho_1+1)\,(\rho_1+2)-\alpha\rho_1},\; b = \frac{2\alpha(1-\alpha)\rho_1\rho_2}{x\,[(\rho_1+1)\,(\rho_1+2)-\alpha\rho_1]}, \text{ and;}$$

$$C = \frac{(1-\alpha)\rho_1}{x}.$$

From the above matrix we find:

$$p_{o2} = \frac{2(1-\alpha)\rho_1}{S} [(1+\rho_1)(2+\rho_1)+\alpha(\rho_2-\rho_1)], \quad (45)$$

$$p_{12} = \frac{2(1-\alpha)\rho_1\rho_2}{S} [(1+\rho_1)(2+\rho_1)+\alpha(\rho_2-\rho_1)], \text{ and}; \quad (46)$$

$$p_{22} = \frac{(1-\alpha)\rho_1}{S} [(1+\rho_2)(2+\rho_2)\{(1+\rho_1)(2+\rho_1)-\alpha\rho_1\}-(1-\alpha)\rho_2(1+\rho_1)$$
$$(2+\rho_1)-2(1+\rho_2)\{(1+\rho_1)(2+\rho_1)+\alpha(\rho_2-\rho_1)\}] \quad (47)$$

where: $S = x\,[(1+\rho_2)(2+\rho_2)\{(1+\rho_1)(2+\rho_1)-\alpha\rho_1\}$
$-(1-\alpha)\rho_2(1+\rho_1)(2+\rho_1)]$ (48)

Using (45) in relations (42) $\rightarrow$ (44) we find more explicit formulas for p_{n1} the probabilities of the first branch.

Also the measures of effectiveness are:

$$p_o = \frac{2}{S} [\alpha\rho_2(1+\rho_2)(2+\rho_2)+(1-\alpha)\rho_1(1+\rho_1)(2+\rho_1-\alpha(1-\alpha)$$
$$(\rho_2-\rho_1)^2] \quad (49)$$

$$L = \frac{2}{S} [\rho_1\rho_2\{\alpha(1+\rho_2)(2+\rho_2)+(1-\alpha)(1+\rho_1)(2+\rho_1)\}+x(1+\rho_2)$$
$$(2+\rho_2)\{(1+\rho_1)(2+\rho_1)-\alpha\rho_1\}-x(1-\alpha)\rho_2(1+\rho_1)(2+\rho_1)-$$
$$2(1+\rho_1)(1+\rho_2)\{\alpha\rho_2(2+\rho_2)+(1-\alpha)\rho_1(2+\rho_1)\}-2\alpha(1-\alpha)$$
$$(\rho_2-\rho_1)\{\rho_2(1+\rho_1)+\rho_1(1+\rho_2)\}],$$

$$\rho_i = \frac{\lambda_i}{\mu}, i = 1, 2. \quad (50)$$

and $L_q = 0$.

Corollary: Let $\alpha = 1$ (or $\alpha\ \frac{1}{2}$ or $\alpha = 0$) we find:

$$p_o = \frac{1}{1+2\rho+2\rho^2}, \text{ and; } L = \frac{2\rho(1+2\rho)}{1+2\rho+2\rho^2}, \rho = \frac{\lambda}{2\mu}$$

which are the measures of the queue: M/M/2/2 with no queue allowed.

3. The Truncated Queue: $H_2/E_2/1/1$

Consider the single–channel truncated hyperexponential interarrival queue having two branches with rates $\alpha\lambda_1$ and $(1-\alpha)\lambda_2$, and service time is Erlangian with rate 2μ. And let us define:

p_{njk} = Probability of n units in the system when the arriving unit is in an arrival branch j and the unit in service is in stage k, $n = 0(1)N$, $j, k = 1, 2$.

and P_{oj} = Probability of no unit in the system when the arriving unit is in the arrival branch j, $j = 1, 2$.

As in White and el [4] we can write the steady–state probability difference equations for the queue: $H_2/E_2/1/1$ are:

$$\rho_1 p_{o1} - p_{112} = 0 \tag{51}$$

$$\rho_2 p_{o2} - p_{122} = 0 \tag{52}$$

$$\alpha\rho_1 p_{o1} + \alpha\rho_2 p_{o2} - [(1-\alpha)\rho_1+1]\, p_{111} + \alpha\rho_2 p_{121} = 0 \tag{53}$$

$$(1-\alpha)\rho_1 p_{o1}+(1-\alpha)\rho_2 p_{o2}+(1-\alpha)\rho_1 p_{111}-(\alpha\rho_2+1)p_{121} = 0 \tag{54}$$

$$p_{111} - [(1-\alpha)\rho_1 + 1]\, p_{112}+\alpha\rho_2 p_{122} = 0 \tag{55}$$

$$p_{121} + (1-\alpha)\rho_1 p_{112} - (\alpha\rho_2 + 1)p_{122} = 0 \tag{56}$$

The previous lemma still has the same form for this queue with the two branches and so we have:

$$p_{o1} + p_{111} + p_{112} = \frac{\alpha\rho_2}{x} \tag{57}$$

$$p_{o2} + p_{121} + p_{122} = \frac{(1-\alpha)\rho_1}{x} \tag{58}$$

To find p_{n1k} the probabilities of the first branch we solve (51), (55) and (57) using the augmented matrix method as:

$$\begin{array}{ccc} p_{o1} & p_{111} & p_{112} \end{array}$$

$$\left[\begin{array}{ccc|c} \rho_1 & 0 & -1 & 0 \\ 0 & 0 & a & -\alpha\rho_2 p_{122} \\ 1 & 1 & 1 & b \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \rho_1 & 0 & -1 & 0 \\ 0 & 1 & a & -\alpha\rho_2 p_{122} \\ 0 & 0 & C & b+\alpha\rho_2 p_{122} \end{array}\right]$$

where $a = -[(1-\alpha)\rho_1+1]$, $b = \frac{\alpha\rho_2}{x}$, and; $C = 1 + \frac{1}{\rho_1} - a$.

From the above matrix we get:

$$p_{112} = \frac{1}{C} (b + \alpha\rho_2 p_{122}) \tag{59}$$

$$p_{111} = -\frac{1}{C} [ab + \alpha\rho_2(a+C)p_{122}] \tag{60}$$

$$p_{01} = \frac{1}{\rho_1 C} (b + \alpha\rho_2 p_{122}) \tag{61}$$

To find p_{n2k} the probabilities of the second branch we solve (52), (56) and (58) using (59) and matrices technique, then we have:

$$\left[\begin{array}{ccc|c} \rho_2 & 0 & -1 & 0 \\ 0 & 1 & d & -(1-\alpha)\rho_1 \frac{b}{C} \\ 1 & 1 & 1 & e \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \rho_2 & 0 & -1 & 0 \\ 0 & 1 & d & -(1-\alpha)\rho_1 \frac{b}{C} \\ 0 & 0 & f & g \end{array}\right]$$

where

$$d=\alpha(1-\alpha) \frac{\rho_1\rho_2}{C} - (\alpha\rho_2+1),\ e = \frac{(1-\alpha)\rho_1}{x},\ f = 1 + \frac{1}{\rho_2} - d$$

$$\text{, and; } g = e + (1-\alpha)\rho_1 \frac{b}{C}.$$

From the above matrix we find:

$$p_{122} = \frac{(1-\alpha)\rho_1\rho_2}{S_1} [(1-\alpha)\rho_1^2+(2+\alpha\rho_2)\rho_1+1] \tag{62}$$

$$p_{121} = \frac{(1-\alpha)\rho_1\rho_2}{S_1} [(1+2\rho_1)(1+\alpha\rho_2)+(1-\alpha)\rho_1^2-\alpha\rho_1(1+\rho_2)^2+\alpha\rho_1\rho_2(1+\rho_2)] \tag{63}$$

$$p_{02} = \frac{(1-\alpha)\rho_1}{S_1} [(1-\alpha)\rho_1^2 + (2+\alpha\rho_2)\rho_1+1) \tag{64}$$

where

$$S_1 = x [(1+\rho_2)^2 \{(1-\alpha)\rho_1^2+2\rho_1+1\} - (1-\alpha)\rho_2^2(1+\rho_1)^2] \tag{65}$$

Using (62) in relations (59) → (61) we find more explicit formulas for p_{n1k} the probabilities of the first branch.

Also the measures of effectiveness are:

$$p_o = p_{o1} + p_{o2} = \frac{1}{S_1} [\alpha\rho_2 \{(1+\rho_2)^2+(1-\alpha)(\rho_1-\rho_2)\} + (1-\alpha)\rho_1 \{(1-\alpha)\rho_1^2+(2+\alpha\rho_2)\rho_1+1\}] \quad (66)$$

$$L = \sum_{n=1}^{1} \sum_{j=1}^{2} \sum_{k=1}^{2} n\, p_{njk} = \frac{1}{S_1} [x(1+\rho_2)^2 \{(1-\alpha)\rho_1^2+2\rho_1+1\} -x(1-\alpha)\rho_2^2(1+\rho_1)^2 -\alpha\rho_2\{(1+\rho_2)^2+(1-\alpha)(\rho_1-\rho_2)\} -(1-\alpha)\rho_1\{(1-\alpha)\rho_1^2+(2+\alpha\rho_2)\rho_1+1\}] \quad (67)$$

and $L_q = 0$

Corollary: Let $\alpha = 1$ (or $\alpha = \frac{1}{2}$ or $\alpha = 0$) we find: $p_o = \frac{1}{2\rho+1}$

$$L = \frac{2\rho}{2\rho+1} .$$

which are the measures of effectiveness of the queue: $M/E_2/1/1$ with no queue allowed.

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