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**On The Sectional Curvatures of The Generalized Ruled Surfaces  
Correspond To Each Other Under The Helical Motion Of Order  $k$  In  
The Euclidean Space  $E^n$**

by

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TURQUIE

**On The Sectional Curvatures Of The Generalized Ruled Surfaces  
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The Euclidean Space  $E^n$**

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**ABSTRACT:**

The purpose of this paper is to present a summary of known results (chapter I, chapter II) and to discuss the sectional curvatures of the fixed and the moving axoids correspond to each other under a helical motion of order  $k$  in  $E^n$ .

Moreover it is proven that the fixed and the moving axoids of  $E^n$  are mapped upon each other by the same values of the sectional curvature.

**I. MOTIONS OF  $E^n$**

A motion of  $E^n$  is described in matrix notation by

$$(1) \quad x = A\bar{x} + c, \quad AA^t = I$$

where  $A^t$  is the transposed of the orthogonal matrix  $A$  and

$$(2) \quad A: J \longrightarrow O(n), \quad c: J \longrightarrow \mathbb{R}^n$$

are functions of differentiability class  $C^r$  ( $r \geq 3$ ) on a real interval  $J$ . Considering a motion as a movement of the space  $\bar{E}$  against the space  $E$  the coordinate vector  $\bar{x}$  in (1) describes a point of the so-called moving space  $\bar{E}$  and  $x$  a point of the so-called fixed space  $E$ .

Let  $\bar{x}$  be fixed in  $\bar{E}$  then (1) defines by (2) a parametrized curve in  $E$  which we call the trajectory curve or path of  $\bar{x}$  under the motion. We get the (trajectory) velocity vector  $\dot{x}$  in the path-point  $x$  from (1) by differentiation (denoted by  $\dot{\phantom{x}}$ ) for  $\bar{x} = 0$  in the form:

$$(3) \dot{x} = B(x-c) + \dot{c}, B = \dot{A}A^{-1}.$$

Since the matrix  $A$  is orthogonal, the matrix  $B$  is skew

$$(4) B + B^t = 0.$$

Therefore only in the case of even dimension is it possible that the determinant  $|B|$  may not vanish. If  $|B(t)| \neq 0$  in  $t \in J$ , we get exactly one solution  $p(t)$  of the equation

$$(5) B(t)(p-c(t)) + \dot{c}(t) = 0.$$

$p(t)$  is the center of the instantaneous rotation of the motion in  $t \in J$  and is called *the pole of the motion* in  $t$ . At a pole  $p$  the velocity vector vanishes by the equation (3). If  $|B|$  does not vanish on  $J$ , by considering the regularity condition of the motion we get a differentiable curve:  $J \rightarrow E$  of poles in the fixed space  $E$ , called *the fixed pole curve*. By (1) there is uniquely determined moving pole curve  $\bar{p}: J \rightarrow \bar{E}$  from the fixed pole curve point to point on  $J$ :  $p(t) = A(t)\bar{p}(t) + c(t)$ .

Müller proved in [4]: "Under the motions the fixed pole curve and the moving pole curve are rolling on each other without sliding. Merely in the case  $n=2$  the motion is determined by a pair of rolling pole curves".

In all other cases (taht means  $|B| = 0$ ), especially for  $n$  odd, we obtain by the rules of Linear Algebra: For every  $t \in J$  there exist an unit vector  $e(t) \in \text{kern } B(t)$  and  $\lambda(t) \in \mathbb{R}$  so that the solutions  $y$  of equation

$$(6) B(t)(y-c(t)) + \dot{c}(t) = \lambda(t)e(t)$$

determined linear subspace  $E_k(t) \subset E^n$  with the dimension  $k=n-\text{rank } B$ .  $E_k(t)$  is *the axis of the instantaneous screw* ( $\lambda(t) \neq 0$ ) of the motion or *the axis of the instantaneous rotation* ( $\lambda(t) = 0$ ) and will be called *the instantaneous axis of the motion* in  $t \in J$ .

If  $|B| = 0$  on the whole interval  $J$  under the regularity conditions we obtain a generalized ruled surface of dimension  $k+1$  in the fixed space  $E$  generated by the instantaneous axes  $E_k(t)$ ,  $t \in J$ , which we call *the fixed axoid  $\Phi$  of the motion*. The fixed axoid  $\Phi$  determines the moving axoid  $\bar{\Phi}$  in the moving space  $\bar{E}$  generator to generator by (1).  $\bar{\Phi}$  and  $\Phi$  are mapped upon each other by the same values of parameter. In this second case Müller proved in [4]: "The axoids  $\Phi, \bar{\Phi}$  of

a motion in  $E^n$  touch each other along every common pair  $E_k(t) \subset \Phi$ ,  $\bar{E}_k(t) \subset \bar{\Phi}$  for all  $t \in J$  by rolling and sliding upon each other under the motion. Such a motion is called an (instantaneously) helical motion of order  $k$  in  $E^n$ , [1].

An helical motion of order  $k$  is a pure rolling for  $\lambda = 0$ . In the special case where the axoids of an helical motion have lines of striction these lines are mapped on to each other point to point.

For the analytical representation of an axoid  $\Phi$  we choose a leading curve  $y$  in the edge resp. central ruled surface  $\Omega \subset \Phi$  transversal to the generator. In [2] it is shown that there exists a distinguished moving orthonormal frame (ONF) of  $\Phi$   $\{e_1, e_2, \dots, e_k\}$  with properties:

- (i)  $\{e_1, e_2, \dots, e_k\}$  is an ONF of the  $E_k(t) \subset \Phi$
- (ii)  $\{e_{m+1}, e_{m+2}, \dots, e_k\}$  is an ONF of the edge space  $K^{k-m}$  resp. the central space  $Z^{k-m} \subset E_k(t)$  in  $E^n$

$$(iii) \dot{e}_\sigma = \sum_{v=1}^k \sigma_{\sigma v} e_v + K_\sigma a_{k+\sigma}, \quad 1 \leq \sigma \leq m$$

$$\dot{e}_{m+\rho} = \sum_{l=1}^m \sigma_{(m+\rho) l} e_l, \quad 1 \leq \rho, \lambda \leq k-m$$

with  $K_\sigma > 0$ ,  $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$ ,  $\sigma_{(m+\rho)(m+\lambda)} = 0$

- (iv)  $\{e_1, e_2, \dots, e_k, a_{k+1}, \dots, a_{k+m}\}$  is an ONF.

A moving ONF of  $\Phi$  with the properties (i) — (iv) is called a principal frame of  $\Phi$ . If  $K_1 > \dots > K_k > 0$  the principal frame of  $\Phi$  is determined up to the signs. By a given principal frame the vectors  $a_{k+1}, \dots, a_{k+m}$  are well defined.

A leading curve  $y$  of an axoid  $\Phi$  is a leading curve of the edge resp. central ruled surface  $\Omega \subset \Phi$  too iff its tangent vector has the form

$$(7) \dot{y} = \sum_{v=1}^k \eta_v e_v + \eta_{m+1} a_{k+m+1}$$

where for  $\eta_{m+1} \neq 0$   $a_{k+m+1}$  is a unit vector well defined up to the sign with the property that  $\{e_1, \dots, e_k, a_{k+1}, \dots, a_{k+m}, a_{k+m+1}\}$  is an ONF of the tangent bundle of  $\Phi$ . One shows:  $\eta_{m+1}(t) = 0$  in  $t \in J$  iff the generator  $E_k(t) \subset \Phi$  contains the edge space  $K_{k-m}(t)$ .

If  $\eta_{m+1} \neq 0$ , we call the  $m$  magnitudes  $\delta_\sigma = \eta_{m+1}/K_\sigma$ ,  $1 \leq \sigma \leq m$ , the *principal parameters of distribution*. These parameters are direct generalizations of the parameter of distribution of the ruled surfaces in  $E^3$  (see [1] and [2]). An axoid with central ruled surface and no principal parameter of distribution ( $m=0$ ) is a  $(k+1)$ -dimensional cylinder [1].

Let  $\Phi$  and  $\bar{\Phi}$  be the corresponding axoids of the given helical motion of order  $k$  in  $E^n$  and  $\{\bar{e}_1, \dots, \bar{e}_k\}$  a principal ONF of the moving axoid  $\bar{\Phi}$ . Then the equation (iii) hold for  $\bar{e}_i$  with barred coefficients.  $\bar{\Phi}$  has the parameter representation on the interval  $J$  by

$$(8) \quad \bar{z}(t, u_1, \dots, u_k) = \bar{y}(t) + \sum_{v=1}^k u_v \bar{e}_v(t), \quad t \in J, u_i \in \mathbb{R}$$

where  $\bar{y}$  is a leading curve of the edge resp. central ruled surface  $\bar{\Omega} \subset \bar{\Phi}$ .

If we set

$$(9) \quad A\bar{e}_v = e_v, \quad 1 \leq v \leq k,$$

then we have the following results (see [1]):

$$(10) \quad B e_v = 0, \quad 1 \leq v \leq k$$

$$(11) \quad A\dot{\bar{e}}_v = \dot{e}_v$$

$$(12) \quad A\bar{a}_{k+\sigma} = a_{k+\sigma}, \quad 1 \leq \sigma \leq m$$

$$(13) \quad \alpha_{\mu\nu} = \bar{\alpha}_{\mu\nu}, \quad K_\sigma = \bar{K}_\sigma > 0, \quad 1 \leq \mu, \nu \leq k, \quad 1 \leq \sigma \leq m$$

$$(14) \quad \eta_{m+1} a_{k+m+1} = \bar{\eta}_{m+1} A\bar{a}_{k+m+1} \quad \text{and} \quad |\bar{\eta}_{m+1}| = |\eta_{m+1}|$$

$$(15) \quad \dot{y} = \lambda e + A\dot{\bar{y}}, \quad \xi_v = \lambda \lambda_v + \bar{\xi}_v, \quad e = \sum_{v=1}^k \lambda_v e_v, \quad \|e\| = 1.$$

Therefore we can give the following theorem:

**Theorem I.1:** Under an helical motion of order  $k$  in  $E^n$  the principal ONFs of the fixed axoid  $\Phi$  and the moving axoid  $\bar{\Phi}$  correspond to each other, the edge spaces (resp. the central spaces) of  $\Phi, \bar{\Phi}$  are mapped on each other point to point. We have  $|\delta_\sigma| = |\bar{\delta}_\sigma|$  for the principal parameters of distribution  $\delta_\sigma$  of  $\Phi$  and  $\bar{\delta}_\sigma$  of  $\bar{\Phi}$ ,  $1 \leq \sigma \leq m$ , [1].

**II. The Sectional Curvatures Of The  $(k+1)$  - Dimensional Ruled Surfaces In The Euclidean Space  $E^n$ .**

Let  $\Phi$  be a  $(k+1)$  -dimensional ruled surface in  $E^n$ . Then  $\Phi$  can be locally represented by

$$(16) \ z(t, u_1, \dots, u_k) = y(t) + \sum_{v=1}^k u_v e_v.$$

If  $\Omega \subset \Phi$  is the  $(k-m+1)$  -dimensional central ruled surface of  $\Phi$ , then the points of  $\Omega \subset \Phi$  can be characterized by the equation

$$(17) \ u_\sigma = 0, \ 1 \leq \sigma \leq m, \text{ (see [3]).}$$

In this chapter we will assume that the leading curve  $y$  of  $\Phi$  is the leading curve of  $\Omega \subset \Phi$ . At each point  $p \in \Phi$  there exist a uniquely determined normal tangential vector  $n$  of  $\Phi$  which is orthogonal to the  $E_k(t) \subset \Phi$  such that

$$(18) \ n = \sum_{i=1}^m u_i K_1(t) a_{k+1}(t) + \eta_{m+1}(t) a_{k+m+1}(t).$$

At  $p \in \Phi$  the section of the each plane  $(e_v, n)$  is called the  $v$ -principal section of  $\Phi$  with respect to the orthonormal frame  $\{e_1(t), \dots, e_k(t)\}$  of  $E_k(t)$ . For the  $v$ -principal section the principal sectional curvature is given by

$$(19) \ K_\sigma(p) = - \frac{(K_\sigma)^2 \left[ \sum_{i=1}^m (u_i K_1)^2 + (\eta_{m+1})^2 - (u_\sigma K_\sigma)^2 \right]}{\left[ \sum_{i=1}^m (u_i K_1)^2 + (\eta_{m+1})^2 \right]^2}$$

and

$$(20) \ K_{m+\rho}(p) = 0, \ 1 \leq \rho \leq k-m, \text{ (see [3]).}$$

Suppose that  $\Phi$  is a  $(k+1)$  -dimensional ruled surface of  $E^n$  and  $\Omega \subset \Phi$  is a central ruled surface of  $\Phi$ . Then at a central point  $z \in \Omega$  the principal sectional curvatures are given by

$$(21) \ K_\sigma(z) = - \frac{1}{\delta_\sigma s}, \ 1 \leq \sigma \leq m$$

and

$$(22) K_{m+\rho}(z) = 0, 1 \leq \rho \leq k-m, \text{ (see [3]).}$$

On the other hand at the point  $z+ue_\sigma$ ,  $u \in \mathbb{R}$  the sectional curvature is

$$(23) K_\sigma(z+ue) = - \frac{\delta_\sigma^2}{[u^2 + \delta_\sigma^2]^2}, 1 \leq \sigma \leq m.$$

Let  $e$  be an arbitrary unit vector in  $E_k(t)$ . Then the sectional curvature of the section  $(e,n)$  can be given by

$$(24) K_z(e,n) = \sum_{\sigma=1}^m (\text{Cos } \varphi_\sigma)^2 K_\sigma(z)$$

$$\text{where } e = \sum_{v=1}^k \text{Cos } \varphi_v e_v.$$

If  $\delta$  is the parameter of distribution for the generator  $\text{Sp}\{e\}$  passing through  $z \in \Omega$ . then

$$(25) K_z(e,n) = - \frac{1}{\delta^2}, \text{ (see [3]).}$$

Moreover at the point  $z \in \Omega \subset \Phi$  the sectional curvature of the arbitrary section  $(e,a)$  is given by

$$(26) K_z(e,a) = \frac{(\text{Cos } \Psi_0)^2}{1 - \langle e,a \rangle^2} K_z(e,n)$$

where

$$(27) a = \frac{\text{Cos } \Psi_0}{\|n\|} n + \sum_{v=1}^k \text{Cos } \Psi_v e_v, \|a\| = 1.$$

In addition at the point  $z+ue_\sigma$  we have

$$(28) [1 - (\text{Cos } \Psi_\sigma)^2] K_{z+vz\sigma}(e_\sigma,a) = (\text{Cos } \Psi_0)^2 K_\sigma(z+ue_\sigma),$$

$z \in \Omega \subset \Phi, 1 \leq \sigma \leq m, \text{ (see [3]).}$

Similarly at the arbitrary point  $p \in \Phi$  with  $p \notin \Omega$  the sectional curvature of the section  $(e,a)$  is given by

$$(29) K_p(e,a) = \frac{(\text{Cos } \Psi_0)^2}{1 - \langle e,a \rangle^2} K_p(e,n)$$

where



$$(30) K_p(e, n) = - \frac{\sum_{\sigma=1}^m (K_{\sigma} \text{Cos } \varphi_{\sigma})^2}{\sum_{l=1}^m (K_l u_l)^2 + (\eta_{m+1})^2} + \frac{\sum_{\sigma, l=1}^m (K_{\sigma} K_l)^2 \text{Cos } \varphi_{\sigma} \text{Cos } \varphi_l u_{\sigma} u_l}{[\sum_{l=1}^m (K_l u_l)^2 + (\eta_{m+1})^2]^2}$$

(see [3]).

Let  $\Phi$  be a  $(k+1)$ -dimensional ruled surface with the central ruled surface  $\Omega \subset \Phi$  and let  $K_{\sigma}(z)$ ,  $1 \leq \sigma \leq m$ , be the principal sectional curvatures of  $\Phi$  at a central point  $z \in \Omega$ . Then the total sectional curvature of  $\Phi$  can be defined by

$$(31) K = \sum_{\sigma=1}^m K_{\sigma}(z)$$

and the mean sectional curvature of  $\Phi$  can be defined by

$$(32) L = \frac{m}{\pi} \sum_{\sigma=1}^m K_{\sigma}(z), \text{ (see [5]).}$$

Moreover the total parameters of distribution of  $\Phi$  and the scalar curvature of  $\Phi$ , respectively, can be defined by

$$(33) D = \sum_{\sigma=1}^m \delta_{\sigma}$$

and

$$(34) R = - 2 L, \text{ (see [5]).}$$

### III. The Sectional Curvatures of The Fixed And Moving Axoids Under The Helical Motion Of Order $k$ In The Euclidean Space $E^n$

In this chapter we will work on the sectional curvatures of the fixed and moving axoids correspond to each other under the helical motion of order  $k$  in  $E^n$ .

Let  $\Phi$  and  $\bar{\Phi}$  be  $(k+1)$ -dimensional the fixed and the moving axoids of  $E^n$  correspond to each other and let  $\Omega \subset \Phi$  and  $\bar{\Omega} \subset \bar{\Phi}$  be  $(k-m+1)$ -dimensional the central ruled surfaces of the axoids. If  $\{e_1(t), \dots, e_k(t)\}$  and  $\{\bar{e}_1(t), \dots, \bar{e}_k(t)\}$  are the fixed and moving principal ONFS of  $\Phi, \bar{\Phi}$  respectively, then the following theorems can be given.

**Theorem III.1:** If  $K_\sigma(z)$  and  $\bar{K}_\sigma(\bar{z})$  are the principal sectional curvatures of  $\Phi$  and  $\bar{\Phi}$  at the points  $z \in \Omega$  and  $\bar{z} \in \bar{\Omega}$ , then

$$K_\sigma(z) = \bar{K}_\sigma(\bar{z}).$$

**Proof:** If we consider the equation (21) for the fixed axoid  $\Phi$  and if we set  $|\delta_\sigma| = |\bar{\delta}_\sigma|$  in this equation we get

$$K_\sigma(z) = -1/\bar{\delta}_\sigma^2, \quad 1 \leq \sigma \leq m.$$

This implies that

$$K_\sigma(z) = \bar{K}_\sigma(\bar{z}), \quad 1 \leq \sigma \leq m.$$

**Theorem III.2:** If  $K_\sigma(z+ue_\sigma)$  and  $\bar{K}_\sigma(\bar{z}+u\bar{e}_\sigma)$  are the principal sectional curvatures of  $\Phi$  and  $\bar{\Phi}$  at the points  $z+ue_\sigma \in \Phi$  and  $\bar{z}+u\bar{e}_\sigma \in \bar{\Phi}$ , then we have

$$K_\sigma(z+ue_\sigma) = \bar{K}_\sigma(\bar{z}+u\bar{e}_\sigma), \quad z \in \Omega \subset \Phi, \bar{z} \in \bar{\Omega} \subset \bar{\Phi}, \quad 1 \leq \sigma \leq m.$$

**Proof:** From the (23) and Theorem I.1. we immediately observe the proof of the theorem.

**Theorem III.3:** At the points  $z \in \Omega \subset \Phi$  and  $\bar{z} \in \bar{\Omega} \subset \bar{\Phi}$

$$K_z(e, n) = \bar{K}_{\bar{z}}(\bar{e}, \bar{n}).$$

**Proof:** Let  $\bar{K}_{\bar{z}}(\bar{e}, \bar{n})$  be the sectional curvature of  $\bar{\Phi}$ . Then by (24) we have

$$(35) \quad K_z(e, n) = \sum_{\sigma=1}^m (\text{Cos } \bar{\varphi}_\sigma)^2 \bar{K}_\sigma(\bar{z}), \quad \bar{z} \in \bar{\Omega}, \bar{e} = \sum_{v=1}^k \text{Cos } \bar{\varphi}_v \bar{e}_v.$$

If we set  $A\bar{e} = e$ , then we find  $e = \sum_{v=1}^k \text{Cos } \varphi_v e_v$ . Hence since  $A$  is an orthogonal transformation and  $A\bar{e}_v = e_v, 1 \leq v \leq k$ , we observe that

$$(36) \cos \varphi_v = \cos \varphi_v, 1 \leq v \leq k.$$

On the other hand since  $\bar{n}$  is an orthogonal vector to the  $\bar{E}_k(t)$ , the vector  $A\bar{n} = n$  is the orthogonal to the  $E_k(t)$  too. If we consider the Theorem III.1. and (36) with the equation (35), we get by (24)

$$(37) \bar{K}_z(\bar{e}, \bar{n}) = K_z(e, n), \bar{z} \in \bar{\Omega}, z \in \Omega.$$

**Corollary III.1:** If  $\delta$  and  $\bar{\delta}$  are the parameters of distribution for the generators  $Sp \{e\}$  and  $Sp \{\bar{e}\}$ , passing through  $z \in \Omega$  and  $\bar{z} \in \bar{\Omega}$ , then  $|\delta| = |\bar{\delta}|$ .

**Proof:** From the Theorem III.3. and because of (25) the proof of the corollary is clear.

Now we would like to observe the sectional curvature of the arbitrary tangential section at the points  $z \in \Omega$  and  $\bar{z} \in \bar{\Omega}$  under the helical motion of order  $k$  in  $E^n$ .

Suppose that

$$(38) \bar{a} = \frac{\cos \bar{\Psi}_0}{\|\bar{n}\|} \bar{n} + \sum_{v=1}^k \cos \bar{\Psi}_v \bar{e}_v, \|\bar{a}\| = 1$$

by (27). If we set  $A\bar{a} = a$ , then since

$$(39) a = \frac{\cos \Psi_0}{\|n\|} n + \sum_{v=1}^k \cos \Psi_v e_v, \|a\| = 1,$$

we obtain

$$(40) \cos \bar{\Psi}_i = \cos \Psi_i, 0 \leq i \leq k.$$

Therefore the following theorem can be given.

**Theorem III.4:**  $\bar{K}_z(\bar{e}, \bar{a}) = K_z(e, a), \bar{z} \in \bar{\Omega}, z \in \Omega.$

**Proof:** If we rewrite the equation (26) for the section  $(\bar{e}, \bar{a})$  we observe that

$$(41) \bar{K}_z(\bar{e}, \bar{a}) = \frac{(\cos \bar{\Psi}_0)^2}{1 - \langle \bar{e}, \bar{a} \rangle^2} \bar{K}_z(\bar{e}, \bar{n}), \bar{z} \in \bar{\Omega}.$$

Moreover if the Theorem III.3. and the equation (40) are considered in this last equation, we get

$$\bar{K}_z(\bar{e}, \bar{a}) = K_z(e, a), \bar{z} \in \bar{\Omega}, z \in \Omega.$$

In addition at points  $\bar{z} + u\bar{e}_\sigma$  and  $z + ue_\sigma$  we have the following corollaries.

**Corollary III.2:**  $\bar{K}_{\bar{z} + u\bar{e}_\sigma}(\bar{e}_\sigma, \bar{a}) = K_{z + ue_\sigma}(e_\sigma, a)$ .

**Proof:** If we consider (28) and (40) together with the Theorem III.2, the proof of the Corollary III.2. is clear.

**Corollary III.3:** At the arbitrary points  $\bar{p} \in \bar{\Phi}$  and  $p \in \Phi$  correspond to each other under the helical motion of order  $k$  we have

$$\bar{K}_{\bar{p}}(\bar{e}, \bar{n}) = K_p(e, n).$$

**Proof:** From (13), (14) and (36) and because of (30) the proof of the Corollary III.3. is clear.

**Corollary II.4:**  $\bar{K}_{\bar{p}}(\bar{e}, \bar{a}) = K_p(e, a)$ ,  $\bar{p} \in \bar{\Phi}$ ,  $p \in \Phi$ .

**Proof:** From the Corollary III.3. and (41) the proof of the Corollary is clear.

**Corollary III.5:** We have the following results.

1.  $K = \bar{K}$
2.  $L = \bar{L}$
3.  $D = \bar{D}$
4.  $R = \bar{R}$ .

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