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by

TANMOY SOM AND R.N. MUKHERJEE

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Some Results On Unique Fixed Points For Mappings On Hausdorff Space

TANMOY SOM* AND R.N. MUKHERJEE

Applied Mathematics Section School of Applied Sciences, Institute of Technology, Banaras Hindu University, Varanasi-221005.

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ABSTRACT:

In the present note we generalize the results of Popa [6] to give some unique fixed point results in Hausdorff space for mappings satisfying the conditions of the type used by Fisher [1], Khan [2,3], Mukherjee and Som [4] and Pal and Maiti [5] in metric spaces.

INTRODUCTION:

Recently Popa [6] gave three unique fixed point results for mappings on a Hausdorff space. In what follows we further give some unique fixed and common fixed point results for the mappings under the conditions similar to that of Fisher [1], Khan [2,3], Mukherjee and Som [4] and Pal and Maiti [5].

Main Results

Theorem 1. Let T be a continuous self mapping of a Hausdorff space X and let f be a continuous function of $X \times X$ into the set of non-negative reals such that

(a) $f(x,y) \neq 0, \forall x \neq y \in X$

(b) Any one of the following two conditions is satisfied for all $x, y \in X$

(i) $f(x,Tx) + f(y,Ty) \leq \alpha f(x,y), 1 \leq \alpha < 2$

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$$\text{or (ii)} \frac{f(x, Tx) f(y, Ty)}{f(Tx, Ty)} + f(Tx, Ty) \leq \alpha f(x, y)$$

with $Tx \neq Ty$ and $1 \leq \alpha < 2$

$$(c) 2f(x, y) \leq f(x, x) + f(y, y), \forall x \neq y \in X,$$

$$(c') f^2(x, y) \leq f(x, x) f(y, y), \forall x \neq y \in X.$$

If for some x_0 , the sequence $\{x_n\} = \{T^n x_0\}$ has a convergent subsequence, then T has a unique fixed point in each case (i) and (ii).

Proof: When case (i) is satisfied we get,

$$f(x_0, Tx_0) + f(x_1, Tx_1) \leq \alpha f(x_0, x_1)$$

$$\text{or, } f(x_1, x_2) \leq (\alpha - 1) f(x_0, x_1) < f(x_0, x_1)$$

Similarly in case (ii) we have

$$\frac{f(x_0, Tx_0) f(x_1, Tx_1)}{f(Tx_0, Tx_1)} + f(Tx_0, Tx_1) (\leq \alpha f(x_0, x_1))$$

$$\text{or, } f(x_1, x_2) \leq (\alpha - 1) f(x_0, x_1) < f(x_0, x_1).$$

By doing the above process again and again, we obtain in both the cases (i) and (ii)

$$f(x_0, x_1) > f(x_1, x_2) > f(x_2, x_3) > \dots \quad (1)$$

which is a monotonic sequence of positive real numbers and therefore converges to some $u \in R^+$ with all its subsequences converging to the same point u . Further let $\{x_{nk}\} \subseteq \{x_n\}$ converges to some $x \in X$. Then since T is continuous, we have

$$Tx = T \lim x_{nk} = \lim Tx_{nk} = \lim x_{nk+1}$$

$$\text{and } T^2 x = T(Tx) = T \lim x_{nk+1} = \lim x_{nk+2}$$

This further gives that

$$\begin{aligned} f(x, Tx) &= f(\lim x_{nk}, \lim x_{nk+1}) \\ &= \lim f(x_{nk}, x_{nk+1}) = u \\ &= \lim f(x_{nk+1}, x_{nk+2}) \\ &= f(Tx, T^2 x). \end{aligned}$$

Now in each of the cases (i) and (ii) we have

$$f(x, Tx) + f(Tx, T^2x) \leq \alpha f(x, Tx)$$

$$\text{or, } f(x, Tx) \leq (\alpha - 1) f(x, Tx)$$

$$\text{and } \frac{f(x, Tx) f(Tx, T^2x)}{f(Tx, T^2x)} + f(Tx, T^2x) \leq \alpha f(x, Tx)$$

$$\text{or, } f(x, Tx) \leq (\alpha - 1) f(x, Tx)$$

which are contradictions unless $x = Tx$ and hence $Tx = x$, i.e. x is a fixed point of T . To prove that fixed point is unique, let $y \neq x$ be another fixed point, then (i) and (ii) gives,

$$f(x, Tx) + f(y, Ty) \leq \alpha f(x, y)$$

$$\text{or, } f(x, x) + f(y, y) \leq \alpha f(x, y) < 2f(x, y)$$

$$\text{and } \frac{f(x, Tx) f(y, Ty)}{f(Tx, Ty)} + f(Tx, Ty) \leq \alpha f(x, y)$$

$$\text{or, } \frac{f(x, x) f(y, y)}{f(x, y)} + f(x, y) \leq \alpha f(x, y)$$

$$\text{or, } f(x, x) f(y, y) \leq (\alpha - 1) f^2(x, y) < f^2(x, y).$$

These contradict the assumptions (c) and (c') respectively and hence the result.

In what follows we give a common fixed point theorem for two mappings on a Hausdorff space.

Theorem 2. Let T_1 and T_2 be two continuous self mappings of a Hausdorff space X and f be a continuous function of $X \times X$ into the set of nonnegative reals such that

$$(a) f(x, y) = f(y, y), \forall x, y \in X$$

$$(b) f(x, y) \neq 0, \forall x \neq y \in X$$

$$(c) \text{ Any one of the following two conditions holds for all } x, y \in X$$

$$(i) f(x, T_1x) + f(y, T_2y) \leq \alpha f(x, y), 1 \leq \alpha < 2$$

$$(ii) \frac{f(x, T_1x) f(y, T_2y)}{f(T_1x, T_2y)} + f(T_1x, T_2y) \leq \alpha f(x, y)$$

with $T_1x \neq T_2y$ and $1 \leq \alpha < 2$.

$$(d) \quad 2f(x, y) \leq f(x, x) + f(y, y), \forall x \neq y \in X,$$

$$(d') \quad f_2(x, y) \leq f(x, x) f(y, y), \forall x \neq y \in X.$$

Further if for some x_0 , the sequence $\{x_n\}$ defined by $T_1x_{2n} = x_{2n+1}$ and $T_2x_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, \dots$, has a convergent subsequence of the type $\{x_{(2p+1)n}\}$, where $p \in N$ is fixed and $n \in N$, then T_1 and T_2 have a unique common fixed point in X .

Proof: From (c) (i) and (ii) it is easy to show that

$$f(x_0, x_1) > f(x_1, x_2) > f(x_2, x_3) > \dots$$

as given in the proof of Theorem 1. Let this sequence $\{f(x_n, x_{n+1})\} \subseteq R^+$ converge to some $u \in R^+$. Further let $\{x_{(2p+1)n}\} \subseteq x_n$ converges to some $x \in X$. Consider the subsequence $\{x_{(2p+1)2n}\}$ of $\{x_{(2p+1)n}\}$. Since T_1 and T_2 are continuous, we have

$$T_1x = T_1 \lim x_{(2p+1)2n'} = \lim x_{(2p+1)2n'+1}$$

$$T_2T_1x = T_2 \lim x_{(2p+1)2n'+1} = \lim x_{(2p+1)2n'+2}$$

Now,

$$\begin{aligned} f(x, T_1x) &= f(\lim x_{(2p+1)2n'}, \lim x_{(2p+1)2n'+1}) \\ &= \lim f(x_{(2p+1)2n'}, x_{(2p+1)2n'+1}) \\ &= \lim f(x_{(2p+1)2n'+1}, x_{(2p+1)2n'+2}) \\ &= f(T_1x, T_2T_1x). \end{aligned}$$

Then from (c) (ii) we have

$$\frac{f(x, T_1x) f(T_1x, T_2T_1x)}{f(T_1x, T_2T_1x)} + f(T_1x, T_2T_1x) \leq \alpha f(x, T_1x)$$

$$\text{or, } f(T_1x, T_2T_1x) \leq (\alpha - 1) f(x, T_1x)$$

$$\text{or, } f(x, T_1x) \leq (\alpha - 1) f(x, T_1x)$$

which is a contradiction unless $x = T_1x$. Case (i) gives the same result and thus $T_1x = x$. Further considering $\{x_{(2p+1)2n'+1}\}$ as a subsequence of $\{x_{(2p+1)n}\}$, we obtain $T_2x = x$ in a similar way and thus x is a common fixed point of T_1 and T_2 . Uniqueness of the fixed point can easily be proved in cases (i) and (ii) under the conditions (d) and (d') respectively in an analogous manner as proved in Theorem 1.

Now we extend our result for a finite set of mappings T_1, T_2, \dots, T_k .

Theorem 3. Let T_1, T_2, \dots, T_k be a set of continuous mappings of a Hausdorff space X into itself and f be a continuous function of $X \times X$ into the set of non-negative reals such that

- (a) $f(x, y) = f(y, x), \forall x, y \in X$
- (b) $f(x, y) \neq 0, \forall x \neq y \in X$
- (c) Any one of the following two conditions is satisfied $\forall x, y \in X$
 - (i) $f(x, T_i x) + f(y, T_{i+1} y) \leq \alpha f(x, y), 1 \leq \alpha < 2$
 - (ii) $\frac{f(x, T_i x) f(y, T_{i+1} y)}{f(T_i x, T_{i+1} y)} + f(T_i x, T_{i+1} y) \leq \alpha f(x, y)$,

with $T_i x \neq T_{i+1} y$ and $1 \leq \alpha < 2$ and $T_{k+1} = T_1$

- (d) $2f(x, y) \leq f(x, x) + f(y, y), \forall x \neq y \in X$
- (d') $f^2(x, y) \leq f(y, y), \forall x \neq y \in X$

If for some $x_0 \in X$, the sequence $\{x_n\}$, defined as

$$x_1 = T_1 x_0, x_2 = T_2 x_1, \dots, x_k = T_k x_{k-1}$$

$$x_{k+1} = T_1 x_k, x_{k+2} = T_2 x_{k+1}, \dots, x_k = T_k x_{2k-1}$$

$$\dots$$

$$x_{nk+1} = T_1 x_{nk}, x_{nk+2} = T_2 x_{nk+1}, \dots, x_{(n+1)k} = T_k x_{(n+1)k-1}$$

for $n = 0, 1, 2, \dots$, has a convergent subsequence of the type $\{x_{(mk+1)n}\}$, where $m, n \in \mathbb{N}$ with some fixed m , then T_1, T_2, \dots, T_k have a unique common fixed point.

Proof: From (c) (i) and (ii) we respectively get

$$f(x_{nk}, T_1 x_{nk}) + f(x_{nk+1}, T_2 x_{nk+1}) \leq \alpha f(x_{nk}, x_{nk+1})$$

$$\text{or, } f(x_{nk+1}, x_{nk+2}) \leq (\alpha - 1) f(x_{nk}, x_{nk+1}) < f(x_{nk}, x_{nk+1})$$

and

$$\begin{aligned} \frac{f(x_{nk}, T_1 x_{nk}) f(x_{nk+1}, T_2 x_{nk+1})}{f(T_1 x_{nk}, T_2 x_{nk+1})} + f(T_1 x_{nk}, T_2 x_{nk+1}) \\ \leq \alpha f(x_{nk}, x_{nk+1}) \end{aligned}$$

$$\text{or, } f(x_{nk+1}, x_{nk+2}) \leq (\alpha - 1) f(x_{nk}, x_{nk+1}) < f(x_{nk}, x_{nk+1})$$

both of which imply that

$$f(x_{n-1}, x_n) > f(x_n, x_{n+1}), n = 1, 2, \dots$$

The rest of the proof that T_1, T_2, \dots, T_k have a unique common fixed point in both the cases (i) and (ii) goes in a similar fashion as that of Theorem 2 by considering consecutive pairs of mappings each time and that each consecutive pair has a unique common fixed point by the same argument. and hence the result.

In what follows we give some more results where the mappings satisfy a rational inequality of the type given by Fisher [1], Khan [2,3].

Theorem 4. Let T be a continuous self mapping of a Hausdorff space X and let f be a continuous function of $X \times X$ into the set of non-negative reals such that

$$(a) f(x, y) \neq 0, \forall x \neq y \in X$$

(b) Any one of the following two conditions is satisfied

$$\forall x, y \in X \text{ with } f(x, Tx) + f(y, Ty) \neq 0$$

$$(i) f(Tx, Ty) \leq c \frac{f(x, Tx) \cdot f(y, Ty)}{f(x, Tx) + f(y, Ty)}, 0 \leq c < 2$$

$$(ii) f(Tx, Ty) \leq \frac{\{b f(x, Tx) + c (y, Ty)\}^2}{f(x, Tx) + f(y, Ty)}$$

where $b, c \geq 0$ with $a = \max \{b, c\}$: $a^2 < \frac{1}{2}$

$$(c) 2f(x, y) \geq f(x, x) + f(y, y), \forall x \neq y \in X.$$

If for some $x_0 \in X$, the sequence $\{x_n\} = \{T_n x_0\}$ has a convergent subsequence, then T has a unique fixed point in each case (i) and (ii).

Proof: The proofs for (1) and the unicity of fixed points go as follows. The rest part of the proof is analogous to that of Theorem 1. We have for case (i)

$$\begin{aligned} f(x_1, x_2) &= f(Tx_0, Tx_1) \\ &\leq \frac{c f(x_0, Tx_0) f(x_1, Tx_1)}{f(x_0, Tx_0) + f(x_1, Tx_1)} \\ &\leq \frac{c f(x_0, x_1) f(x_1, x_2)}{f(x_0, x_1) + f(x_1, x_2)} \end{aligned}$$

$$\text{or, } f(x_0, x_1) + f(x_1, x_2) \leq c f(x_0, x_1)$$

$$\text{or, } f(x_1, x_2) \leq (c-1) f(x_0, x_1) < f(x_0, x_1)$$

Similarly $f(x_2, x_3) < f(x_1, x_2)$ and so on.

For showing unicity in this case let $Tx = x$ and $Ty = y$,

$x \neq y$. Then from (i)

$$f(x, y) = f(Tx, Ty)$$

$$\leq \frac{c f(x, Tx) f(y, Ty)}{f(x, Tx) + f(y, Ty)}$$

$$\leq \frac{2 f(x, x) \cdot f(y, y)}{f(x, x) + f(y, y)}$$

$$\leq \frac{\{f(x, x) + f(y, y)\}^2 - \{f(x, x) - f(y, y)\}^2}{2 [f(x, x) + f(y, y)]}$$

$$\text{or, } f(x, y) < \frac{f(x, x) + f(y, y)}{2}$$

$$\text{or, } 2 f(x, y) < f(x, x) + f(y, y)$$

which contradicts the condition (c) of the theorem and so $x = y$. In case (ii) we have

$$f(x_1, x_2) = f(Tx_0, Tx_1)$$

$$\leq \frac{\{b f(x_0, Tx_0) + c f(x_1, Tx_1)\}^2}{f(x_0, Tx_0) + f(x_1, Tx_1)}$$

$$< \frac{\{a f(x_0, x_1) + a f(x_1, x_2)\}^2}{f(x_0, x_1) + f(x_1, x_2)}$$

$$\text{or, } f(x_1, x_2) < \frac{a^2}{1-a^2} f(x_0, x_1) < f(x_0, x_1)$$

and similarly $f(x_2, x_3) < f(x_1, x_2)$ and so on.

To prove uniqueness let $Tx = x$ and $Ty = y$,

$x \neq y$. Then from (ii), we have

$$f(x, y) = f(Tx, Ty)$$

$$\leq \frac{\{a f(x, Tx) + a f(y, Ty)\}^2}{f(x, Tx) + f(y, Ty)}$$

$$f(x,y) < \frac{1}{2} [f(x,x) + f(y,y)]$$

which again contradicts condition (c) and therefore $x = y$.

Next we state two unique common fixed point results for two mappings T_1, T_2 and for the set of mappings T_1, T_2, \dots, T_k satisfying a rational inequality similar to that of the previous Theorem 4.

Theorem 5. Let T_1 and T_2 be two continuous self mappings of a Hausdorff space X and let f be a continuous function from $X \times X$ into the set of nonnegative real numbers such that

$$(a) f(x,y) = f(y,x), \forall x, y \in X$$

$$(b) f(x,y) \neq 0, \forall x \neq y \in X$$

(c) Any one of the following two conditions holds for all $x, y \in X$ with $f(x, T_1x) + f(y, T_2y) \neq 0$

$$(i) f(T_1x, T_2y) \leq c \frac{f(x, T_1x) f(y, T_2y)}{f(x, T_1x) + f(y, T_2y)}, 0 \leq c < 2$$

$$(ii) f(T_1x, T_2y) \leq \frac{\{b f(x, T_1x) + c f(y, T_2y)\}^2}{f(x, T_1x) + f(y, T_2y)},$$

where $b \geq 0, c \geq 0$ with $a = \max \{b, c\}$: $a^2 < \frac{1}{2}$

$$(d) 2f(x,y) \geq f(x,x) + f(y,y), \forall x \neq y \in X.$$

Further if for some x_0 , the sequence $\{x_n\}$, defined as $x_{2n+1} = T_1x_{2n}$ and $x_{2n+2} = T_2x_{2n+1}$, $n = 0, 1, \dots$, has a convergent subsequence of the type $\{x_{(2p+1)n}\}$, where $p \in N$ is fixed and $n \in N$, then T_1, T_2 have a unique common fixed point.

Theorem 6. Let T_1, T_2, \dots, T_k be a set of continuous mappings of a Hausdorff space X into itself and let f be a continuous function from $X \times X$ into the set of nonnegative reals such that

$$(a) f(x,y) = f(y,x), \forall x, y \in X$$

$$(b) f(x,y) \neq 0, \forall x \neq y \in X$$

(c) Any one of the following two conditions is satisfied for all $x, y \in X$ with $f(x, T_1x) + f(y, T_{k+1}y) \neq 0$

$$(i) f(T_ix, T_{i+1}y) \leq c \frac{f(x, T_ix) f(y, T_{i+1}y)}{f(x, T_ix) + f(y, T_{i+1}y)}, 0 \leq c < 2$$

$$(ii) f(T_i x, T_{i+1} y) \leq \frac{\{b f(x, T_i x) + c f(y, T_{i+1} y)\}^2}{f(x, T_i x) + f(y, T_{i+1} y)}, b \geq 0,$$

$c \geq 0$ with $a = \max \{b, c\}$: $a^2 < \frac{1}{2}$ and $T_{k+1} = T_1$

(d) $2 f(x, y) \geq f(x, x) + f(y, y), \forall x \neq y \in X.$

If for some $x_0 \in X$, the sequence $\{x_n\}$ as defined in Theorem 3 has a convergent subsequence of the type $\{x_{(m_k+1)n}\}$, where $m, n \in N$ with m fixed, then T_1, T_2, \dots, T_k have a unique common fixed point.

The proofs of the above two theorems follow from that of Theorem 4 combining with that of Theorem 2 and Theorem 3 respectively.

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