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On Truncated Poisson Queue With Balking, Reneging and Heterogenity.

by

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On Truncated Poisson Queue With Balking, Reneging and Heterogenity.

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ABSTRACT

In this paper the author treats a poissonian queue with the following three properties: Balking, Reneging and Heterogenity. A modified queue discipline to the classical one 1 st in 1 st out (FIFO) is used with a more general condition. In fact this paper is a continuation and more generalization to the works done by many researchers such as: Abou – El – Ata, Singh, Krishna – moorthi and others.

1. INTRODUCTION

In fact this work was early studied by many researchers in case of only one property. Krishnamoorthi [3] considered a poisson queue with heterogenity and two alternative queue disciplines one with slight and the other with a great modification of the classical one (FIFO). Singh [5] analyzed a Markovian queue with balking and heterogenity with a modified queue discipline but different than those of Krishnamoorthi. Abou – El – Ata [1] considered a Markovian queue with both balking and heterogenity with a modified queue discipline of both Singh [5] and Krishnamoorthi [3]. In fact he had generalized both of their works.

In this paper the author adds a more third property $(n-1)\alpha$ due to the reneging factor. This means that the author treats a truncated poisson queue (capacity N) with balking, reneging and heterogenity. A modified queue discipline of both Singh and Krishnamoorthi is used with a more general condition. In fact and really this work is a generalization of most of the previous works.

2. Analyzing of the problem

Consider the truncated two-channels queue: M /M /2 /N with balking, reneging and heterogenity. The arrivals are poisson distributed with mean rate λ and the service times are exponentially distributed with mean rates μ_i, $i = 1,2$. The queue discipline considered is:

(i) If both channels are free, the top unit of the queue goes to channel I with probability π_1 or to channel II with probability π_2 $(\pi_1 + \pi_2 = 1)$.

(ii) If one of them is free, the top unit goes to it.

(iii) If both channels are busy the top unit waits until the 1 st one is free and then goes to it. The new coming unit joins the queue with probability β (i.e the unit balks with prob. $1-\beta$). Any unit in the queue leaves it with probability $[(n-2)\alpha]$, $n = 2\ (1)N$, {which means it renges with prob. $(n-2)\ \alpha$}.

Let us define the probabilities:

$p_0(t) = p_{00}(t) =$ prob. {there is no unit in the system at time t}

$p_{10}(t) =$ prob. {there is one unit in channel I at time t}

$p_{01}(t) =$ prob. {there is one unit in channel II at time t}

$p_n(t) =$ prob. {there are n units in the system at time t, $n = 2\ (1) \ldots$}

Also: $p_1(t) = p_{10}(t) + p_{01}(t)$, and; $p_2(t) = p_{11}(t)$

As in the usual arguments of the δ-technique the steady-state probability difference equations are:

$$-\lambda p_0 + \mu_1 p_{10} + \mu_2 p_{01} = 0 \qquad n = 0 \qquad (1)$$

$$\left.\begin{aligned} &-(\lambda + \mu_1)\, p_{10} + \mu_2 p_{11} + \lambda\pi_1 p_0 = 0 \\ &-(\lambda + \mu_2)\, p_{01} + \mu_1 p_{11} + \lambda\, \pi_2 p_0 = 0 \end{aligned}\right\} \quad n = 1 \qquad (2)$$

$$-(\beta\lambda + \mu)\, p_2 + (\mu + \alpha)\, p_3 + \lambda p_1 = 0 \qquad n = 2 \qquad (3)$$

$$-[\beta\lambda + \mu + (n-2)\alpha]P_n + [\mu + (n-1)\alpha P_{n+1} + \beta\lambda P_{n-1} = 0, \quad N-1 \geq n > 2 \qquad (4)$$

$$-[\mu + (N-2)\,\alpha]\, p_N + \beta\lambda\, p_{N-1} = 0 \qquad n = N \qquad (5)$$

Where: $\mu = \mu_1 + \mu_2$.

Adding the 1st four equations in (1), (2) and (3)

We get:

$$(\mu + \alpha)\, p_3 - \beta\lambda\, p_2 = 0 \qquad (6)$$

From (3) and (6) we have:

$$0 = (\mu + \alpha)\, p_3 - \beta\lambda\, p_2 = \mu\, p_2 - \lambda p_1$$

i.e $$p_2 = \rho\, p_1 \tag{7}$$

From: (2) and (7) we find:

$$p_{10} = \frac{\lambda\,[\lambda + (\mu_1 + \mu_2)\,\pi_1]}{\mu_1\,[2\,\lambda + \mu_1 + \mu_2]}\, p_0, \quad \text{and;}$$

$$p_{01} = \frac{\lambda\,[\lambda + (\mu_1 + \mu_2)\,\pi_2]}{\mu_2\,[2\lambda + \mu_1 + \mu_2]}\, p_0$$

Therefore: $p_1 = \Delta\, p_0$ (8)

Where: $$\Delta = \frac{\lambda\,(\mu_1 + \mu_2)\,[\lambda + \mu_1\,\pi_2 + \mu_2\,\pi_1]}{\mu_1\,\mu_2\,(2\lambda + \mu_1 + \mu_2)}$$

$$= \frac{\lambda\,[\lambda + \mu_1\,\pi_2 + \mu_2\,\pi_1]}{\mu_1\,\mu_2\,(2\,\rho + 1)}, \quad \rho = \frac{\lambda}{\mu} \tag{9}$$

From (4) and (6) we can get:

$$[\mu + (n-1)\,\alpha]\, p_{n+1} - \beta\,\lambda p_n = [\mu + (n-2)\,\alpha]\, p^{n-} \beta\lambda\, p_{n-1}$$
$$= \ldots = (\mu + \alpha)\, p_3 - \beta\lambda\, p_2 = 0$$

i.e $$p_n = \frac{\beta\lambda}{\mu + (n-2)\,\alpha}\, p_{n-1}, \quad n = 2\,(1)\,N-1 \tag{10}$$

Using (7) in (10) we find:

$$p_n = \frac{\rho\,(\beta\gamma)^{n-2}}{(\delta + 1)_{n-2}}\, p_1\,, \quad n = 2\,(1)\,N-1 \tag{11}$$

Where: $\gamma = \dfrac{\lambda}{\alpha}$ and $\delta = \dfrac{\mu}{\alpha}$

Also from (5): $$p_N = \frac{\beta\gamma}{\delta + (N-2)}\, p_{N-1} \tag{12}$$

Thus from (8), (11), and (12) we have:

$$p_n = \frac{\Delta\rho\,(\beta\gamma)^{n-2}}{(\delta+1)_{n-2}}\, p_0 \quad n = 2\,(1)\,N \tag{13}$$

where p_0 could be found from the boundary condition: $\sum_{n=o}^{N} p_n = 1$
as follows:

$$p_0 + p_1 + \sum_{n=2}^{N} p_n = 1$$

$$p_0 + \Delta\, p_0 + \Delta\, \rho\, p_0 \sum_{n=0}^{N} \frac{(\beta\gamma)^{n-2}}{(\delta + 1)_{n-2}} = 1$$

$$\text{i.e } p_0 + \Delta\, p_0 + \Delta\, \rho\, p_0 \sum_{n=0}^{N-2} \frac{(1)_n\, (\beta\gamma)^n}{n!\, (\delta + 1)_n} = 1$$

Thus: $p_0^{-1} = 1 + \Delta + \Delta\, \rho\, {}_1F_1\, (-(N-2);\, \delta + 1;\, -\beta\gamma)$ (14)

where Δ is given in (9).

The measures of effectiveness are:

$$E\,(n) = \sum_{n=1}^{N} n\, p_n = \Delta\, p_0 + \Delta\rho\, p_0 \sum_{n=2}^{N} \frac{n\, (\beta\gamma)^{n-2}}{(\delta + 1)_{n-2}}$$

$$= \Delta p_0 + \Delta\rho\beta\gamma p_0 \frac{d}{d(\beta\gamma)} \{{}_1F_1\, (-(N\text{-}2);\, \delta + 1,\, -\beta\gamma)\}$$

$$+ 2\, \Delta\, \rho\, p_0\, {}_1F_1\, (-(N-2);\, \delta + 1;\, -\beta\gamma) \qquad (15)$$

$$\text{and } E\,(m) = \sum_{n=2}^{N} (n-1)\, p_n$$

$$= \Delta\, \rho\, \beta\, \gamma\, p_0 \frac{d}{d\,(\beta\gamma)} \{\, {}_1F_1\, (-(N-2);\, \delta + 1;\, \beta\gamma)\}$$

$$+ \Delta\, \rho\, p_0\, {}_1F_1\, (-(N-2);\, \delta + 1,\, -\beta\gamma) \qquad (16)$$

Corollary (1): Let $\alpha = 0$, this is the truncated poisson queue with both balking and heterogenity. Thus from (13) we get

$$p_n = \Delta\, \rho\, (\beta\, \rho)^{n-2} \cdot p_0, \qquad n = 2\, (1)\, N \qquad (17)$$

Where: $p_0^{-1} = 1 + \dfrac{\Delta}{1 - \beta\rho}\, [1 + (1 - \beta)\, \rho - \rho\, (\beta\rho)^{N-1}]$ (18)

and Δ is given in (9).

The measures of effectiveness in this case are:

$$E(n) = \Delta\, p_0\, (1 + x) \text{ and } E(m) = \Delta\, p_0\, x \tag{19}$$

$$\text{where } x = \frac{\beta\, \{1 - (\beta\rho)^{N-1}\}}{1 - \beta\rho} + \frac{\rho\, \{1 - (N-1)(\beta\rho)^{N-2} + N(\beta\rho)^{N-1}\}}{(1 - \beta\rho)^2} \tag{20}$$

Corollary (2): Let $\alpha = 0$ and $N \to \infty$, we have the poisson queue with both balking and heterogenity which is Abou – El – Ata [1].

Thus from (7), (18), (19) and (20) we find:

$$p_n = \Delta\, \rho\, (\beta\rho)^{n-2} \cdot p_0, \tag{21}$$

$$p_0 = 1 + \frac{\Delta\, [1 + (1 - \beta)\, \rho]}{1 - \beta\rho}$$

$$= \frac{\lambda\, [\lambda + \mu_1 \pi_2 + \mu_2 \pi_1]\, [1 + (1 - \beta)\, \rho]}{\mu_1 \mu_2 (2\rho + 1)(1 - \beta\rho)} \tag{22}$$

$$E(n) = \Delta\, p_0 \cdot \left[1 + \frac{\rho}{1 - \beta\rho} + \frac{\rho}{(1 - \beta\rho)^2}\right] \tag{23}$$

$$\text{and } E(m) = \frac{\Delta\, \beta\rho^2\, p_0}{(1 - \beta\rho)^2} \tag{24}$$

Corollary (3): Let $\alpha = 0$, $\beta = 1$ and $N \to \infty$, we obtain a poisson queue with heterogenity.

Thus from (13), (14), (15) and (16) we have:

$$p_n = \Delta\, \rho^n\, p_0\,, \; n \geq 1 \tag{25}$$

$$p_0 = \frac{\mu_1 \mu_2 (2\rho + 1)(1 - \rho)}{\mu_1 \mu_2 (2\rho + 1)(1 - \rho) + \lambda\, (\lambda + \mu_1 \pi_2 + \mu_2 \pi_1)}, \tag{26}$$

$$E(n) = \Delta\, p_0 \left[1 + \frac{\rho}{1 - \rho} + \frac{\rho}{(1 - \rho)^2}\right] \tag{27}$$

$$\text{and } E(m) = \frac{\Delta\, \rho^2\, p_0}{(1 - \rho)^2} \tag{28}$$

This is Krishnamoorthi's work [3].

Corollary (4): Let $\alpha = 0$, $N \to \infty$ and $\pi_1 = \pi_2$,

we get:

$$p_n = \Delta \rho \, p_0 (\beta\rho)^{n-2}, \; n = 2 \, (1) \ldots \qquad (29)$$

$$p_0 = \frac{\mu_1\mu_2 (2\rho + 1)(1 - \beta\rho)}{\mu_1\mu_2 (2\rho + 1)(1 - \beta\rho) + \lambda(\lambda + \mu_2)\,[1 + (1 - \beta)\rho\,]} \qquad (30)$$

$$E(n) = -\Delta \, p_0 \left[1 + \frac{\rho}{1 - \beta\rho} + \frac{\rho}{(1 - \beta\rho)^2}\right] \qquad (31)$$

$$\text{and } E(m) = \frac{\Delta \beta \rho^2}{(1 - \beta\rho)^2} p_0 \qquad (32)$$

Here Δ is different from that of Abou - El - Ata's [1] in Corollary (2) because ($\pi_1 = \pi_2$). This is Singh's work [5]. Finally we can deduce Gumble's work [2] if we let $\alpha = 0$, $\beta = 1$, $N \to \infty$ and $\pi_1 = \pi_2 = \frac{1}{2}$. Also Saaty's work [4] if we let $\alpha = 0$, $\beta = 1$, $N \to \infty$ and

$$\pi_1 = \frac{\mu_1}{\mu_1 + \mu_2}, \; \pi_2 = \frac{\mu_2}{\mu_1 + \mu_2}.$$

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