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Matrix Transformations on Cesâro Difference Sequence Spaces

by

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Matrix Transformations on Cesâro Difference Sequence Spaces

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SUMMARY

In a recent paper, [3], we have defined the Cesàro difference sequence spaces

$$C_p = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right|^p < \infty,\ 1 \leq p < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_k) : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right| < \infty,\ n \geq 1 \right\}$$

and determined some matrix classes related to these spaces. In this paper, we go on to give some matrix classes of the same type.

1. INTRODUCTION

In [3], we have defined the Cesáro difference sequence spaces

$$C_p = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right|^p < \infty,\ 1 \leq p < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_k) : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right| < \infty,\ n \geq 1 \right\}$$

and showed that the inclusion

$$ces_p \subset X_p \subset C_p$$

is strict for $1 \leq p \leq \infty$, where $\Delta x_k = x_k - x_{k+1}$, $(k = 1, 2, \ldots)$, and ces_p and X_p are sequence spaces defined by

$$\text{ces}_p = \left\{ x = (x_k) : \| x \|_p = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} | x_k | \right)^p \right)^{1/p} < \infty, \right.$$

$$\left. 1 \leq p < \infty \right\}$$

$$X_p = \left\{ x = (x_k) : \| x \|_p = \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty, \right.$$

$\left. 1 \leq p < \infty \right\}$ respectively, ([4], [1]).

Further, the inclusion $l_p \subset \text{ces}_p \subset X_p \subset C_p$ is also strict for $1 < p < \infty$, where

$$l_p = \left\{ x = (x_k): \sum_{k=1}^{\infty} | x_k |^p < \infty, 1 \leq p < \infty \right\}.$$

The matrix transformations on Cesáro Sequence spaces of a non-absolute type are given in [2].

In [3], it is showed that the sequence spaces C_p, $1 \leq p < \infty$, and C_∞ are Banach spaces under the certain norms. Moreover, the Generalized Köthe- Toeplitz duals of these space are determined and some related matrix classes are given, [3].

In this paper, we go on to determine some matrix classes related to these spaces.

2. PRELIMINARIES

If we define the operator $S: C_p \longrightarrow C_p$; $1 \leq p \leq \infty$; by $x \longrightarrow S(x) = (0, x_2, x_2 \ldots)$, then S is a bounded linear operator on C_p with $\| S \| = 1$. Furthermore,

$$S(C_p) = \{ x = (x_k) : x \in C_p, x_1 = 0 \} \subset C_p$$

is a subspace of C_p, $1 \leq p \leq \infty$, [3].

The following result may be found in [3].

LEMMA. 2.1. Let σ be defined on $S(C_p)$, $1 \leq p \leq \infty$, by $\sigma(x) = (\sigma_n(x))$ where

$$\sigma_n(x) = \frac{1}{n} \sum_{k=1}^{n} \Delta x_k = \frac{-x_{n+1}}{n}, \ (n = 1, 2 \ldots).$$

Then σ is an one-to-one bounded linear transformation from $S(C_p)$ onto the sequence space l_p with the operator norm 1.

We note that, if $A = (a_{nk})$ is an infinite matrix of complex numbers $a_{nk}(n, k=1,2, ...)$ and if

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k, \quad (n=1,2, \ldots),$$

exists for each n and for all $x \in X$ and also $(A_n(x)) \in Y$, then the matrix $A = (a_{nk})$ defines a matrix transformation from X into Y where X, Y are any two subspaces of the space of complex sequences. Now, let (X,Y) be the set of all infinite matrices $A = (a_{nk})$ which map the sequence space X into the sequence space Y.

Throughout the paper, when an infinite matrix $A = (a_{nk})$ is given, we associate three other matrices B,D and F with A as follows:

$$B = (b_{nk}) = (k.a_{n,k+1})$$

$$D = (d_{nk}) = \frac{1}{n}(a_{1k} - a_{n+1,k})$$

$$F = (f_{nk}) = (k.d_{n,k+1})$$

for all n,k.

3. MAIN RESULTS

In this paragraph, we shall give some matrix transformations of the sequence space C_p, $(1 \leq p \leq \infty)$.

The matrices of classes (C_p, E), $1 < p \leq \infty$, have been determined in Theorem.5.1, [3], where E denotes one of the sequence spaces l_∞ and c, namely the linear space of bounded and convergent sequences, respectively.

Now, we begin to determine the matrices of classes (C_1, E).

THEOREM. 3.1. Let $A = (a_{nk})$ be a matrix such that $(a_{n1}) \in E$. Then $A \in (C_1, E)$ if and only if $B \in (l_1, E)$.

Since the theorem may be proved as in ([3; Theorem 5.1] we omit the proof.

THEOREM. 3.2. Let $1 \leq p < \infty$ and $A = (a_{nk})$ be a matrix such that $(a_{n1}) \in l_p$. Then $A \in (C_1, l_p)$ if and only if $B \in (l_1, l_p)$.

Proof. Necessity: If $A \in (C_1, l_p)$, then, the series $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each n whenever $x \in C_1$. Further $(A_n x)) \in l_p$. Particularly, for every $x \in S(C_1) \subset C_1$, the series

$$A_n(x) = \sum_{k=2}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} a_{n,k+1}\, x_{k+1}$$

is convergent. Now, using Lemma 2.1. we get

$$(1) \quad A_n(x) = - \sum_{k=1}^{\infty} k\, a_{n,k+1}\, t_k$$

where $k\, t_k = - x_{k+1}$, $(k = 1,2,...)$. That is to say thay, each series in the statement (1) is convergent for all sequences $t = (t_k)$ belonging to l_1, and so $B \in (l_1, l_p)$.

Sufficiency: If $x = (x_k) \in C_1$, then

$$x_k = \begin{cases} x_1, & k = 1 \\ y_k, & k \geq 2 \end{cases}$$

where $y = (y_k) \in S(C_1)$. Hence for all $x \in C_1$, we can write, formally

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k = a_{n1}\, x_1 + \sum_{k=2}^{\infty} a_{nk}\, y_k$$

$$= a_{n1}\, x_1 + \sum_{k=1}^{\infty} a_{n,k+1}\, y_{k+1}.$$

By Lemma 7.1, for every $x \in C_1$

$$A_n(x) = a_{n1}\, x_1 - \sum_{k=1}^{\infty} k\, a_{n,k+1}\, t_k$$

where $k\, t_k = - y_{k+1}$, $(k=1,2,...)$. Therefore,

$$(2) \quad A_n(x) = a_{n1}\, x_1 - B_n(t)$$

for every $x \in C_1$ and for all $t \in l_1$. Now, the hypothesis gives that

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k, \text{ on } C_1,$$

exists for each n. On the other hand, applying the Minkowski inequality to the statement (2), we get

$$\left(\sum_{n=1}^{\infty} |A_n(x)|^p\right)^{1/p} \leq |x_1| \left(\sum_{n=1}^{\infty} |a_{n1}|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |B_n(t)|^p\right)^{1/p}$$

This yields $A \in (C_1, l_p)$. Now, the proof is completed.

THEOREM. 3.3. Let the first row of the matrix $A = (a_{nk})$ be a finite sequence. Then $A \in (C_1, C_p)$ if and only if $D \in (C_1, l_p)$ where $1 \leq p \leq \infty$.

Proof. We recall that a sequence $x = (x_n)$ is called finite if and only if there exists $k \in \mathbb{N}$ such that $x_n = o$ for all $n \geq k$.

Let $1 \leq p < \infty$ and suppose that $A \in (C_1, C_p)$ Then, for every $x \in C_1$, $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ exists and $(A_n(x)) \in C_p$. Accordingly

$$\sum_{i=1}^{\infty} \left| \frac{1}{i} (A_1(x) - A_{i+1}(x)) \right|^p < \infty$$

and therefore

$$\sum_{i=1}^{\infty} \left| \frac{1}{i} \sum_{k=1}^{\infty} (a_{1k} - a_{i+1}) x_k \right|^p < \infty$$

for every $x \in C_1$. This gives that $D \in (C_1, l_p)$.

Conversely, let $D \in (C_1, l_p)$. Then, the series

$$(3) \quad D_i(x) = \sum_{k=1}^{\infty} d_{ik} x_k = \frac{1}{i} \sum_{k=1}^{\infty} (a_{1k} - a_{i+1,k}) x_k$$

converges for every $x \in C_1$ and for each i. Since the sequence (a_{1k}) $(k = 1,2,\ldots)$, is finite, (3) implies that

$$A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$$

is convergent for every $x \in C_1$ and for each i. Moreover, $(D_i(x)) \in l_p$ yields that $(A_i(x)) \in C_p$. The case $p = \infty$ can also be examined in a similar way.

COROLLARY.3.4. Let $A=(a_{nk})$ be a matrix such that its first row is a finite sequence and first coloumn is in C_p, $1 \leq p < \infty$. Then $A \in (C_1, C_p) \Leftrightarrow D \in (C_1, l_p) \Leftrightarrow F \in (l_1, l_p)$.

The proof is an immediate consequence of Theorem 3.2 and 3.3

The following corollary holds by Theorem 3.1 and 3.3.

COROLLARY. 3.5.. Let $A = (a_{nk})$ be a matrix such that its first row is a finite sequence and first coloumn is in C_∞. Then

$A \in (C_1, C_\infty) \Leftrightarrow D \in (C_1, l_\infty) \Leftrightarrow F \in (l_1, l_\infty)$.

The following theorems 3.6 and 3.7 can be obtained by a similar argument as in Theorem 3.2. and Theorem 3.3, respectively.

THEOREM. 3.6.. Let $A = (a_{nk})$ be a matrix such that $(a_{n1}) \in l_2$. Then

$A \in (C_2, l_2)$ if and only if $B \in (l_2, l_2)$.

THEOREM. 3.7. Let $A = (a_{nk})$ be a matrix as in Theorem 3.3. Then $A \in (C_2, C_2)$ if and only if $D \in (C_2, l_2)$.

COROLLARY. 3.8. Let $A = (a_{nk})$ be a matrix such that its first row is a finite sequence and first coloumn is in C_2. Then

$A \in (C_2, C_2) \Leftrightarrow D \in (C_2, l_2) \Leftrightarrow F \in (l_2, l_2)$.

The proof follows from Theorem 3.6 and 3.7.

THEOREM. 3.9. Let $1 < p \leq \infty$. Then $A \in (C_p, l_1)$ if and only if

(i) $(a_{n1}) \in l_1$

(ii) $B \in (l_p, l_1)$.

Proof. If $A \in (C_p, l_1)$, then the series $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ is convergent and $(A_n(x)) \in l_1$, for each n, and for all $x \in C_p$. If we just put $x = (1,0,0,...) \in C_p$, then the necessity of (i) is trivial. Write again the statement (1) for $x \in S(C_p) \subset C_p$, then the necessity of (i) is trivial. If we write again the statement (1) for $x \in S(C_p) \subset C_p$, then the necessity of (ii) is obvious.

For the converse, write $x = (x_k) \in C_p$ as follows:

$$x_k = \begin{cases} x_1, & k = 1 \\ y_k, & k \geq 2 \end{cases}$$

where $y = (y_k) \in S(C_p)$. Now, reconsider the statement (2) for $x \in S(C_p)$. And therefore sufficiency holds by (i) and (ii). Hence the proof is completed.

THEOREM. 3.10. Let $1 < p \leq \infty$ and $A = (a_{nk})$ be a matrix as in Theorem 3.3. Then

$A \in (C_p, C_1)$ if and only if $D \in (C_p, l_1)$.

The theorem is proved exactly as in Theorem 3.3.

Theorem 3.9 and 3.10 give the following

COROLLARY. 3.11. Let $1 < p \leq \infty$ and $A = (a_{nk})$ be a matrix as in Theorem 3.3. Then

$A \in (C_p, C_1) \Leftrightarrow D \in (C_p, l_1) \Leftrightarrow F \in (l_p, l_1)$ and $(a_{n1}) \in C_1$.

We remark that the matrices of classes (l_1, E); (l_1, l_p), $1 \leq p < \infty$; (l_2, l_2); (l_p, l_1), $1 < p \leq \infty$, are well-knownn in summability theory, (see [5]).

ÖZET

[3] te

$$C_p = \left\{ x = (x_k): \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right|^p < \infty, 1 \leq p < \infty \right\}$$

ve

$$C_\infty = \left\{ x = (x_k): \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right| < \infty, n \geq 1 \right\}$$

Cesâro fark dizi uzaylarını tanımlamış ve bu uzaylarla ilgili bazı matris sınıflarını belirlemiştik. Bu çalışmada da, bu uzaylarla ilgili bazı matris sınıflarını belirlemeye devam ettik.

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