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A LA MEMOIRE D'ATATÜRK AU CENTENAIRE DE SA NAISSANCE

On The Order Of Approximation To A Function By Generalized Gauss Weierstrass Singular Integrals

by

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DEDICATION TO ATATÜRK'S CENTENNIAL

Holding the torch that was lift by Atatürk in the hope of advancing our Country to a modern level of civilization, we celebrate the one hundredth anniversary of his birth. We know that we can only achieve this level in the fields of science and technology that are the wealth of humanity by being productive and creative. As we thus proceed, we are conscious that, in the words of Atatürk, "the truest guide" is knowledge and science.

As members of the Faculty of Science at the University of Ankara we are making every effort to carry out scientific research, as well as to educate and train technicians, scientists, and graduates at every level. As long as we keep in our minds what Atatürk created for his Country, we can never be satisfied with what we have been able to achieve. Yet, the longing for truth, beauty, and a sense of responsibility toward our fellow human beings that he kindled within us gives us strength to strive for even more basic and meaningful service in the future.

From this year forward, we wish and aspire toward surpassing our past efforts, and with each coming year, to serve in greater measure the field of universal science and our own nation.

On The Order Of Approximation To A Function By Generalized Gauss Weierstrass Singular Integrals

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ABSTRACT

In this paper the authors have determined the degree of approximation of a certain class of functions by Generalized Gauss-Weierstrass Singular integrals. This class of functions contains those belonging to Lip α and Lip (α, p) due to Hardy and Littlewood [1].

1. INTRODUCTION

Let f (x) be an integrable function belonging to $L_1 (-\infty, +\infty.)$ We determine the order of approximation by Generalized Gauss-Weierstrass singular integrals of f (x) defined as follows:

For $s > 0$,

$$(1.1)\ L(x, \xi; s) = \frac{s}{2\Gamma\frac{1}{s}} \cdot \frac{1}{\xi^{1/s}} \int_{-\infty}^{\infty} f(x+t)\, e^{-\frac{|t|^s}{\xi}}\, dt, \quad \xi \to o+$$

We also define the following notations.

$$(1.2)\ \varnothing_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

and

$$(1.3)\ \Phi(t) = \int_0^t \varnothing_x(v)\, dv.$$

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DEFINITION (1.1): A function, integrable L, is said to belong to the class k (t), where k (t) is a positive increasing function, $\frac{k(t)}{t}$ is decreasing and such that;

(a) $k(xy) = k(x)\,k(y)$.

(b) $|f(x+t) - f(x)| = 0\,(k(t))$.

We notice that,

(i) If $k(t) = t^{\alpha}$, $0 < \alpha < 1$, then our class reduces to Lip α.

(ii) If $k(t) = t^{\alpha - \frac{1}{p}}$ and $f(x) \in L^p$ $(p > 1)$, then our class reduces to Lip (α, p). (Hardy and Littlewood [1]).

(iii) If $k(t) = \Psi(t)\, t^{-\frac{1}{p}}$, $\Psi(t)$ is a positive increasing function and $f(x) \in L^p$ $(p > 1)$, then our class reduces to the class $(\Psi(t), p)$. (Huzoor, H. Khan [2]).

2 The main purpose of this paper is to determine, by taking into account, the more general class k (t), the order of approximation for Generalized Gauss-Weierstrass singular integrals of f (x).

We state and prove the following theorems:

Theorem (2.1): If f (x) belongs to $L_1(-\infty, +\infty)$ and for $u \to 0$,

$$\int_0^u [f(x+t) + f(x-t) - 2f(x)]\, dt = 0\,(u^2 k(u))$$

then,

$$|L(x; \xi; s) - f(x)| = 0\,(\xi^{1/s} k(\xi^{1/s})),\ \xi \to 0+$$

provided k (t) satisfies the following,

$$(2.1)\quad \begin{cases} \left(\dfrac{t_o}{\xi^{1/s}}\right)^2 \dfrac{k(t_o)}{k(\xi^{1/s})}\, e^{-\left(\frac{|t_o|}{\xi^{1/s}}\right)^s} = o(1), \text{ and} \\[2ex] \dfrac{d}{dt}\,(t^2 k(t)) = 0\,(t\,k(t)). \end{cases}$$

Where, we break the interval $(0, \infty)$ in to, $(0, t_o)$ and (t_o, ∞).

Theorem (2.2): If f (x) belongs to $L_1(-\infty, +\infty)$ and for $u \to 0$,

$\int_0^u \lambda(t)\,dt = 0\,(u^2 k(u))$, and

$$\lambda(t) = \int_{-\infty}^{\infty} |f(x+t) + f(x-t) - 2f(x)|\,dx,$$

then,

$$\int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)|\,dx = 0\,(\xi^{1/s} k(\xi^{1/s})),\ \xi \to 0+$$

Under the same conditions (2.1).

Proof of Theorem:

Proof of Theorem (2.1). Since

$$L(x;\xi;s) - f(x) = \frac{s}{2\Gamma\frac{1}{s}} \cdot \frac{1}{\xi^{1/s}} \int_0^{\infty} \varnothing_x(t)\, e^{-\frac{|t|^s}{\xi}}\,dt.$$

Then on dividing both sides by $\xi^{1/s}\ k\ (\xi^{1/s})$, we have

$$\frac{1}{\xi^{1/s} k(\xi^{1/s})}[L(x;\xi;s) - f(x)] = \frac{s}{2\Gamma\frac{1}{s}\,\xi^{2/s} k(\xi^{1/s})} \int_0^{t_o} e^{-\frac{|t|^s}{\xi}}\,d\Phi(t)$$

$$+ \frac{s}{2\Gamma\frac{1}{s}\,\xi^{2/s} k(\xi^{1/s})} \Big[\int_{t_o}^{\infty} f(x+t)\, e^{-\frac{|t|^s}{\xi}}\,dt + \int_{t_o}^{\infty} f(x-t)\, e^{-\frac{|t|^s}{\xi}}\,dt$$

$$- \int_{t_o}^{\infty} 2f(x)\, e^{-\frac{|t|^s}{\xi}}\,dt \Big].$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say,}$$

$$I_1 = \frac{s}{2\Gamma\frac{1}{s}\,\xi^{2/s} k(\xi^{1/s})} \Big[e^{-\frac{|t_o|^s}{\xi}}\,\Phi(t_o) - \int_0^{t_o} \Phi(t)\,d\,(e^{-\frac{|t|^s}{\xi}}) \Big].$$

$$= I_1' + I_1'', \text{ say.}$$

$$I'_1 = \frac{s}{2\Gamma\frac{1}{s}} \left(\frac{t_o}{\xi^{1/s}}\right)^2 \frac{k(t_o)}{k(\xi^{1/s})} e^{-\left(\frac{|t_o|}{\xi^{1/s}}\right)^s} \frac{\Phi(t_o)}{t_o^2\, k(t_o)}$$

$$= o(1).$$

Next, since $\Phi(t) = 0(t^2 k(t))$.

$$I''_1 = \frac{s}{2\Gamma\frac{1}{s}\, \xi^{2/s} k(\xi^{1/s})}\, 0\left[\int_0^{t_o} t^2 k(t)\, d\left(e^{-\frac{|t|^s}{\xi}}\right)\right].$$

$$= 0\left[\left(\frac{t_o}{\xi^{1/s}}\right)^2 e^{-\left(\frac{|t_o|}{\xi^{1/s}}\right)^s} \frac{k(t_o)}{k(\xi^{1/s})}\right] +$$

$$0\left[\int_0^{t_o}\left(\frac{t}{\xi^{1/s}}\right) \frac{k(t)}{k(\xi^{1/s})} e^{-\left(\frac{|t|}{\xi^{1/s}}\right)^s} d\left(\frac{|t|}{\xi^{1/s}}\right)\right].$$

$$= o(1) + 0(1)$$

$$= 0(1) \text{ for } \xi \to 0+$$

Therefore,

$$I_1 = I'_1 + I''_1$$

$$= o(1) + 0(1)$$

$$= 0(1)$$

Now,

$$|I_2| \leqq \frac{s}{2\Gamma\frac{1}{s}\, \xi^{2/s} k(\xi^{1/s})} e^{\frac{|t_o|^s}{\xi}} \int_{t_o}^{\infty} |f(x+t)|\, dt.$$

$$\leqq \frac{M.\, s}{2\Gamma\frac{1}{s}\, t_o^2\, k(t_o)} \left(\frac{t_o}{\xi^{1/s}}\right)^2 \frac{k(t_o)}{k(^{1/s})} e^{-\left(\frac{|t_o|}{\xi^{1/s}}\right)^s}.$$

$$= o(1), \text{ for } \xi \to 0+$$

similarly, $I_3 = o(1)$.

Again,

$$|I_4| \leqq \frac{s\,|f(x)|}{\Gamma\frac{1}{s}\ \xi^{2/s}\,k(\xi^{1/s})} \int_{t_o}^{\infty} e^{-\frac{|t|^s}{\xi}}\,dt.$$

$$\leqq \frac{s\,|f(x)|}{\Gamma\frac{1}{s}\ t_o\,k(t_o)} \int_{t_o}^{\infty} \frac{t\,k(t)}{\xi^{2/s}\,k(^{1/s})}\, e^{-\frac{|t|^s}{\xi}}\,dt.$$

$$\leqq \frac{s\,|f(x)|}{\Gamma\frac{1}{s}\ t_o\,k(t_o)} \int_{\frac{t_o}{\xi^{1/s}}}^{\infty} \frac{(\xi^{1/s}\,v)\ k(\xi^{1/s})\,k(v)\,e^{-v^s}\,\xi^{1/s}\,dv}{\xi^{2/s}\,k(\xi^{1/s})}$$

on putting $\frac{t^s}{\xi} = v^s$.

$$= \frac{s\,|f(x)|}{\Gamma\frac{1}{s}\ t_o\,k(t_o)} \int_{\frac{t_o}{\xi^{1/s}}}^{\infty} v\,k(v)\,e^{-v^s}\,dv$$

$$= o(1) \text{ as } \int_o^{\infty} v\,k(v)\,e^{-v^s}\,dv < +\infty$$

Adding the bounds,

$$I_1 + I_2 + I_3 + I_4 = 0(1) + o(1) + o(1) + o(1)$$
$$= 0(1).$$

Hence

$$|L(x;\xi;s) - f(x)| = 0(\xi^{1/s}\,k(\xi^{1/s})).$$

which completes the proof of theorem (2.1).

Proof of Theorem (2.2):

We have,

$$\frac{1}{\xi^{1/s}\,k(\xi^{1/s})} \int_{-\infty}^{\infty} |L(x;\xi;s) - f(x)|\,dx$$

$$\leqq \frac{s}{2\Gamma\frac{1}{s}\ \xi^{1/s}\,k(\xi^{2/s})} \int_{-\infty}^{\infty} dx \int_0^{\infty} |\varnothing_x(t)|\, e^{-\frac{|t|^s}{\xi}}\,dt.$$

$$= \frac{s}{2\Gamma\frac{1}{s}\ \xi^{2/s}\ k(\xi^{1/s})} \int_0^\infty e^{-\frac{|t|^s}{\xi}}\, dt \int_{-\infty}^{\infty} |\varnothing_x(t)|\, dx.$$

$$= \frac{s}{2\Gamma\frac{1}{s}\ \xi^{2/s}\ k(\xi^{1/s})} \int_0^\infty e^{-\frac{|t|^s}{\xi}}\, \lambda(t)\, dt.$$

We can write, as in the proof of Theorem (2.1),

$I_1 = o(1)$ for $\xi \to 0+$

and,

$$I_2 + I_3 + I_4 = \frac{s}{2\Gamma\frac{1}{s}\ \xi^{2/s}\ k(\xi^{1/s})} \int_{t_o}^\infty \lambda(t)\, e^{-\frac{|t|^s}{\xi}}\, dt$$

$$\leqq \frac{s.\ M}{2\Gamma\frac{1}{s}\ t_o\, k(t_o)} \int_{t_o}^\infty \frac{t.\ k(t)}{\xi^{2/s}\ k(\xi^{1/s})}\ e^{-\frac{|t|^s}{\xi}}\, dt.$$

$= 0(1),\ \xi \to 0+.$

Therefore,

$$\begin{aligned} I_1 + I_2 + I_3 + I_4 &= 0(1) + 0(1) \\ &= 0(1). \end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} |L(x;\ \xi;\ s) - f(x)|\, dx = 0(\xi^{1/s}\ k(\xi^{1/s})).$$

Which completes the proof of Theorem (2. 2)

Now, using the condition $0(t\,k(t))$ on $\varnothing_x(t)$, we have the following theorems.

Theorem (2.3) : If $f(x)$ belongs to $L_1(-\infty, +\infty)$ and for $u \to o$,

$$\int_o^u [f(x+t) + f(x-t) - 2f(x)]\ dt = 0(u\, k(u)),$$

then

$$|L(x;\ \xi;\ s) - f(x)| = 0(k(\xi^{1/s})),\ \xi \to 0$$

provided that,

$$(2.2)\quad \left(\frac{t_o}{\xi^{1/s}}\right) \frac{k(t_o)}{k(\xi^{1/s})} \quad e^{-\left(\frac{|t_o|}{\xi^{1/s}}\right)^s} = o(1), \text{ and}$$

$$\frac{d}{dt}(t\ k(t)) = 0\ (k(t))$$

where, we break the interval (o, ∞) in t_o, (o, t_o) and (t_o, ∞).

Theorem (2.4) : If $f(x) \in L_1(-\infty, +\infty)$ and for $u \to o$,

$\int_o^u \lambda(t)\,dt = 0\,(u\ k(u))$, and

$\lambda(t) = \int_{-\infty}^{\infty} |f(x+t) + f(x-t) - 2f(x)|\ dx,$

then

$$\int_{-\infty}^{\infty} |L(x; \xi; s) - f(x)|\ dx = 0\ (k(\xi^{1/s})),\ \xi \to 0$$

under the same conditions (2.2).

The proof of theorems (2.3) and (2.4) are similar to those of (2.1) and (2.2) respectively.

Remark It may be remarked that on giving different values of k (t), we get some interesting results.

(i) If $k(t) = t^{\alpha}$, we have,

$$|L(x; \xi; s) - f(x)| = 0\,(\xi^{\frac{1+\alpha}{s}}).$$

(ii) If $k(t) = t^{\alpha - \frac{1}{p}}$, we get,

$$|L(x; \xi; s) - f(x)| = 0\,(\xi^{\frac{1}{s} + \frac{\alpha}{s} - \frac{1}{sp}}).$$

(iii) If $k(t) = \Psi(t)\, t^{-\frac{1}{p}}$, we obtain

$$|L(x; \xi; s) - f(x)| = 0\,(\xi^{\frac{1}{s} - \frac{1}{sp}}\ \Psi(\xi^{\frac{1}{s}})).$$

The last special case has been considered by Huzoor H. Khan [3].

ÖZET

Bu çalışmada, genelleştirilmiş Gauss Weierstrass singüler integralleri yardımıyla Hardy ve Littlewood [1] tarafından tanımlanan Lip ve Lip (α,p) sınıflarına ait olan fonksiyonları içeren bir fonksiyon sınıfının yaklaşım derecesi belirlenmiştir.

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