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# Invariant Means And Multiplicative Matrices


by

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# Invariant Means And Multiplicative Matrices

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**ABSTRACT**

Schaefer [7] has introduced the concept of $\sigma$ conservative and $\sigma$ regular matrices and obtains necessary and sufficient conditions to characterize these classes of matrices. In the present paper authors have defined $(c, V\sigma)\beta$-matrices and $(\omega_p, V_\sigma)\beta$ and obtain necessary and sufficient conditions to characterize them.

## 1. INTRODUCTION.

Let $l_\infty$, c and $c_o$ be the Banach spaces of bounded, convergent and null sequences $x = \{x_k\}$ with usual norm $\|x\| = \sup_k |x_k|$. Let $\sigma$ be a mapping of the set of positive integers into itself. A continuous linear function $\varnothing$ on $l_\infty$ is said to be an invariant mean or a $\sigma$ - mean if and only if (i) $\varnothing(x) \geq o$ for all n, (ii) $\varnothing(e) = 1$, where $e = \{1,1,1, .....\}$, and (iii) $\varnothing\{x_{\sigma(n)}\} = \varnothing(x)$ for all $x \in l_\infty$. Throughout this paper we deal only with mappings which are one to one such that $\sigma^m(n) \neq n$ for all n and m where $\sigma^m(n)$ denotes the $m^{th}$ iterate of the mapping at n.

For such mappings, every $\sigma$-mean extends the limit functional on c. (see Raimi [6]). Consequently, $c \subset V_\sigma$ where $V_\sigma$ is the set of bounded sequences all of whose $\sigma$-means are equal. When $\sigma(n) = n+1$, $V\sigma$ is the set'f'of almost convergent sequences (see Lorentz [1]).

P. Schaefer [7] has defined the concept of $\sigma$-conservative and $\sigma$-regular and characterized these classes of matrices i.e. $(c, V_\sigma)$, $(c, V_\sigma)_{reg}$. Recently Mursaleen [5] characterized the classes of $(w_p, V_\sigma)$ and $(w_p, V_\sigma)_{reg}$. Eizen and Laush [1] obtained necessary and sufficient conditions to characterize the

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matrices of the class $(c,f)_\beta$. In the sequel, the object of this paper is to obtain necessary and sufficient conditions to characterize the matrices of the classes of $(c,V_\sigma)_\beta$ and $(w_p, V_\sigma)_\beta$, which will fill up a gap in the existing literature.

2. Preliminaries:

If $p_k$ is real such that $p_k > o$ and $\sup_k p_k < \infty$, we define (Maddox [4])

$$w(p) = \{ x : \frac{1}{n} \sum_{k=1}^{n} | x_k - l |^{pk} \to 0 \text{ for some } l \}.$$

If E is a subset of S, the space of complex sequences, then we shall write $E^+$ for the generalized Köthe - Toeplitz dual of E, i. e.

$$E^+ = \{a : \sum_k a_k x_k \text{converges for every } x \in E\}$$

If $o < p_k \leq 1$, then $w^+(p) = M$, where

$$M = [\, a : \sum_{r=o}^{\infty} \max_r \{ (2^r N^{-1})^{1/p_k} | a_k | \} < \infty \text{ for some}$$

integer $N > 1$ and $\max_r$ is the maximum taken over $2^r \leq k$

$< 2^{r+1}]$ (see Lascarides and Maddox [2]).

When $p_k = p \ \forall\ k$, we have $w(p) = w_p$.

If x is a topological linear space we shall denoto $x^*$ the continuous dual of x, i.e. the set of all continuous linear functionals on x.

If $x = \{x_n\}$, write $Tx = \{x_{\sigma(n)}\}$. It is easy to show that the set $V_\sigma$ can be characterized as the set of all bounded sequences for which $\lim_m (x + Tx + .... + T^m x) / (m + 1)$ exists in $l_\infty$

and has the form L e, $L = \sigma - \lim x$.

Throughout this paper we shall use the notation a (n,k) to denote the element $a_{nk}$ of the matrix A, for $m \geq 0$, we have

$$(Ax + TAx + .... + T^m Ax) / (m + 1)$$

$$= \{ \sum_k [a(n,k) + a(\sigma(n), k) + ..... + a(\sigma^m(n), k)] x_k / (m + 1) \}_{n=1}^{\infty}$$

where $\sigma^m(n)$ denotes the $m^{th}$ iterate of $\sigma$ at n. For every $n \geq 1$, put

$$T_{mn}(x) = t_{mn}(Ax) = \sum_{k=1}^{\infty} \sum_{j=0}^{m} a\,(\sigma^j(n),\ k)\ x_k / (m+1)$$

$$= \sum_k \alpha\,(n,k,m)\ x_k$$

where

$$\alpha\,(n,k,m) = \frac{1}{m+1} \sum_{j=o}^{m} a\,(\sigma^j(n),\ k).$$

3. $(c, V_\sigma)_\beta$– matrices:

The notion of $(c, f)_\beta$ – matrices (Eizen and Laush [1] can be generalized as follows.

DEFINITION 1. An infinite matrix A is said to be $(c, V_\sigma)_\beta$ –matrix if and only if it is $\sigma$ – conservative and and $\sigma$ – lim Ax =lim $\beta$ x for all $x \in c$.

THEOREM 1. $A \in (c, V_\sigma)_\beta$ if and only if

$$\| A \| = \sup_n \{ \sum_k | a_{nk} | \} < +\infty \tag{3.1}$$

$$a_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \in V_\sigma \text{ with } \sigma\text{-limit zero for each } k \tag{3.2}$$

$$a = \{ \Sigma\ a_{nk}\}_{n=1}^{\infty} \in V_\sigma \text{ with } \sigma\text{-limit } \beta. \tag{3.3}$$

PROOF: Soppose that $A \in (c, V\sigma)_\beta$. If $x \in c_o$, then $Ax \in V_\sigma \subset l_\infty$. It follows from the proof of Theorem 1 [7] that $\| A \| < +\infty$. Define $e = \{1,1,1,....\}$ and $e^k = \{0,0,0,1$ ($k^{th}$–place), $0,0\}$. Since $Ae = a$ and $Ae^k = a_{(k)}$, (3.2) and (3.3) must hold i.e., $\sigma$-lim $Ae^k = 0 = \sigma$ – lim $a_{(k)}$ and $\sigma$– lim $Ae = \beta = \sigma$–lim a.

Conversely, let us suppose that (3.1), (3.2) and (3.3) must hold. Let $x \in c$, we have from the proof of Theorem 1 [7] that

$$\| T_{mn} \| < +\infty.$$

Therefore the hypothesis of Theorem 1 [7] holds with $u_k = o$ and $u = \beta$ and so A is $\sigma$ – conservative.

Therefore

$$\lim_m T_{mn}(x) = \lim\ [T_n(e) - \sum_k T_n(e^k)\ ]\ x + \sum_k x_k\ T_n(e^k)$$

$$= \lim \beta\ x,$$

where

$$\lim_m T_{mn}(e) = T_n(e) = \beta \text{ and } \lim_m T_{mn}(e^k) = T_n(e^k) = 0.$$

So that $A \in (c, V_\sigma)_\beta$.

4. $(w_p, V_\sigma)_\beta$– matrices.

DEFINITION 2. An infinite matrix A is said to be $(w_p, V_\sigma)_\beta$–matrix if and only if it is $(w_p, V_\sigma)$ – matrices and the $\sigma - \lim Ax = \lim \beta x$ for all $x \in w_p$.

THEOREM 2. Let $1 \leq p < \infty$, then $A \in (w_p, V_\sigma)_\beta$ if and only if

$$D(A) = \sup_m \sum_r 2^{r/p} \left( \sum_r | \alpha(n,k,m) |^q \right)^{1/q} < +\infty$$

for every n, where $1/p + 1/q = 1$,

$$a_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \in V\sigma \text{ with the } \sigma - \text{limit zero for each k,} \quad (4.2)$$

$$a = \{\sum_k a_{nk}\}_{n=1}^{\infty} \in V\sigma \text{ with the } \sigma \text{–limit } \beta. \quad (4.3)$$

PROOF : Suppose that $A \in (\omega P, V_\sigma\text{-})_\beta$. Define $e = \{1,1,1, .....)$ and $e^k = \{o,o,o,1$ ($k^{th}$ place), $o,o,\}$. Since $Ae = a$ and $Ae^k = a_{(k)}$, (4.2) and (4.3) must hold, i.e. $\sigma - \lim Ae^k = 0 = \sigma - \lim a_{(k)}$ and $\sigma - \lim Ae = \beta = \sigma - \lim a$. For the necessity of (4.1), suppose that $T_{m,n}(x) = \sum_r [\alpha\ (n,k,m)\ x_k$ exists for each $x \in \omega p$ then for each m and $r \geq o$, define $f_{rm} = \sum_r \alpha\ (n,k,m)\ x_k$. Then $\{f_{rm}\}_m$ is a sequence of continuous linear functionals on $w_p$, since

$$| f_{rm}(x) | \leq \left( \sum_r | \alpha\ (n,k,m) |^q \right)^{1/q} \left( \sum_r | x_k |^p \right)^{1/P}$$

$$\leq 2^{r/p} \left( \sum_r | \alpha\ (n,k,m) |^q \right)^{1/q} \| x \|.$$

It follows ( [4], corollary on pp. 114) that for each m

$$\lim_j \sum_{r=o}^{j} f_{rm}(x) = T_{mn}(x)$$

is in the dual space $w^*_p$, whence there exists $K_{mn}$ such that

$$|T_{mn}(x) | \leq K_{mn} \| x \|. \quad (4.4)$$

For each m we take any integer $j > o$ and defining $x \in w_p$ as in ( [4], Theorem 7), we have

$$\sum_{r=o}^{j} 2^{r/p} \ ( \sum_{r} \mid \alpha \ (n,k,m) \mid^{q})^{1/q} \leq K_{mn}$$

whence, for each m

$$\sum_{r=0}^{\infty} 2^{r/p} \ ( \sum_{r} \mid \alpha \ (n,k,m) \mid^{q})^{1/q} \leq K_{mn} \leq \infty . \qquad (4.5)$$

Now, since $T_{mn}(x)$ is absolutely convergent, we have

$$\mid T_{mn}(x) \mid \ \leq \sum_{r=p}^{\infty} 2^{r/p} \ ( \sum_{r} \mid \alpha \ (n,k,m) \mid^{q})^{1/q} \| \ x \ \|,$$

so that

$$K_{mn} \leq \sum_{r=o}^{\infty} 2^{r/p} \ ( \sum_{r} \mid \alpha \ (n,k,m) \mid^{q})^{1/q}, \qquad (4.6)$$

By virtue of (4.5) and (4.6),

$$K_{mn} = \sum_{r=o}^{\infty} 2^{r/P} \ ( \sum_{r} \mid \alpha \ (n,k,m) \mid^{q})^{1/q}$$

Finally, (by Theorem 11 ( [4]*), pp. 114) for every n, the existence of $\lim_{m} T_{mn}(x)$ on $w_p$implies that

$$\sup_{m} K_{mn} = \sup_{m} \sum_{r=o}^{\infty} 2^{r/p} \ ( \sum_{r} \mid \alpha \ (n,k,m) \mid^{q})^{1/q} < \infty ,$$

which is (4.1).

Conversely, let us suppose that conditions (4.1), (4.2) and (4.3) hold and $x \in w_p$. Now

$$\mid T_{mn}(x) \mid \ \leq \sum_{r=o}^{\infty} \ \sum_{r} \mid \alpha \ (n,k,m) \ x_k \mid$$

$$\leq \sum_{r=o}^{\infty} \ ( \sum_{r} \mid \alpha \ (n,k,m) \mid^{q})^{1/q} ( \sum_{r} \mid x_k \mid^{p})^{1jp}$$

$$\leq D \ (A) \ \| \ x \ \|.$$

Therefore $T_{mn}(x)$ is absolutely and uniformly convergent for each m. Also $\sum_{r=o}^{\infty} 2^{r/p} \left( \sum_{r} | u_k |^q \right)^{1/q} < \infty$, and by Holder's inequality $\sum_{r} u_k x_k < \infty$. Therefore the hypothesis of Theorem 2 holds with $u_k = o$ and $u = \beta$ and so $A \in (w_p, V_\sigma)$.

Therefore,

$$\lim_{m} T_{mn}(x) = \lim \, [T_n(e) - \sum_{k} T_n(e^k) \,] \, x + \sum_{k} x_k T_n(e^k)$$

$$= \lim \beta \, x.$$

where $\lim_{m} T_{mn}(e) = T_n(e) = \beta$ and $\lim T_{mn}(e^k) = T_n(e^k) = 0$ so that $A \in (\omega_p, V\sigma)_\beta$.

5. We have the follwing corollaries.

It we take $\beta = 1$ in Theorem 1, we have

Corollary 5.1: (see Schaefer [7] ). The matrix A is $\sigma$ – regular if and only if

$$\| A \| < + \infty$$

$a_{(k)} \in V\sigma$ with $\sigma$ – limit zero each k, and (5.1.2)

$a \in V\sigma$ with $\sigma$ – limit $+ 1$ (5.1.3)

If we take $\beta = 1$ in Theorem 2, we get

Corollary 5.2: (see Mursaleen [5].) Let $o < p < \infty$, Then $A \in (w_p, V\sigma)_{reg}$ if and only if conditions (4.1), (4.2) with $\sigma$–limit = o and (4.3) with $\sigma$ limit + 1.

If we put $\sigma(n) = n+1$ in Theorem 1, we have

Corollary 5.3: (see Eizen and Laush [1]). $A \in (c,f)_\beta$ – matrix with multiplier $\beta$ if and only if

$$\| A \| < + \infty \tag{5.3.1}$$

$$\lim_{n} \alpha \, (n,k,m) = 0 \text{ (uniformly in m,k, fixed)} \tag{5.3.2}$$

$$\lim_{n} \sum_{k} \alpha \, (n,k,m) = \beta \text{ (uniformly in m)} \tag{5.3.3}$$

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## ÖZET

$\sigma$ konservatif ve $\sigma$regüler matris kavramları Schaefer tarafından tamamlanmış olup bu matris sınıflarını karakterize eden gerek ve yeter koşullar elde edilmiştir. (7). Bu çalışmanın amacı ise $(c, V\sigma)$ $(\omega_p, V\sigma)_\beta$ – matrislerini tanımlamak ve bunları karakterize edecek gerek ve yeter koşulları elde etmektir.

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