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On The Chebyshev Approximation by A + B* log (1 + CX)

by

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1

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On The Chebyshev Approximation by $A + B^* \log (1 + CX)$

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ABSTRACT

Previous studies on the Chebyshev approximation are enlightened, and the best Chebyshev approximation proved to be $A+B^*\log(1+CX)$ on $[0,\alpha]$ and it is generalized with the help of new concepts.

INTRODUCTION

The most general approximation problem, first presented in 1970 by Barrodal [1], can be express shortly as the following:

On the assumption that X is a topologic space and C(X) a set of bounded and continious functions (have real and complex values) on space X, C(X) space can be set up by norm

$$\|g\| = \sup\{ |g(x)| ; x \in X \}$$

Let P be a parameter space and F approximation function in C(X) corresponding an element A of parameter space P such as $F(A,.) = F[A]$. There is an element, F [A], for f which is in C(X) such that

$$\rho(f,X) = \inf \{ \|f - F[A^*]\|; A \in P \}$$

with the condition of

$$\rho(f,X) = \|f - F[A^*]\|$$

then A is called "best parameter" and the function $F[A^*]$ "best approximation" to f on X. Searching A^* is the essential of Chebyshev problem.

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Solution of Chebyshev approximation problem is carried out by means of varying X, F and P. The conditions hold in for the solution of Chebyshev problem are important.

G. Meinardus and Schwedt [2] found out important theorems in 1964 which are used for the best approximation in Chebyshev problem. Then many scientists have studied on Chebyshev aproximation problem under various conditions [3]. C.B. Dunham [4], [5] proved that the best approxition would be A+B*log (1+CX) on [0, α].

In our study we set up new lemmas, theorems and definitions in order to enlighten the obscurities in previous studies and to prove the best Chebyshev approximation to be A+B* log (1+CX) on [0, α]. Furthermore, we have generalized it by means of new concepts.

EXTENSIVE SOLUTION OF CHEBYSHEV APPROXIMATION BY A+B*log(1+CX)

Topologic concepts are invariant under an homomorphism. [-1,+1] is homomorph to [0,α] so we can use [-1,+1] instead of [0,α].

Let C([-1,+1]) be the space of defined and numerical functions on [-1,+1] with norm

$$\|g\| = \sup\{|g(x)|;\ -1 \leq x \leq +1\}$$

and with the condition

$$P = \{A: A = (a_1,a_2,a_3) \in \mathcal{R}^3\}$$

Consider the existence of approximation function F, corresponding to element f on the same space, C([-1,+1]). Let the approximation function has the form of

$$F(A,x) = a_1+a_2\log\ (1+a_3x)$$

for an element A of a selected parameter space, P. When $\|a_3\|\geq 1$, $\|F(A,.)\|$ goes infinity so that the parameter a_3 satisfies

$$-1 < a_3 < +1$$

After selecting an approximation function F as above, finding element A* for which $\|f-F(A,.)\|$ is minimum, gives solution of

Chebyshev problem. Such an element A^* is called "best parameter" and $F(A^*,.)$ "best approximation" to f.

We can put approximation functions of the type

$$F(A,x) = a_1 + a_2 \log(1 + a_3 x)$$

into two groups:

1. **Constant approximation**

Constant approximation is such approximation functions that correspond to parameters $A = (a_1, 0, a_3)$ or $A = (a_1, a_2, 0)$. Really in this case $F(A,x) = a_1$.

2. **Non-constant approximation**

Now $a_2 \neq 0$ and $a_3 \neq 0$, that is $a_2 a_3 \neq 0$. In this case approximation function is evidently unique.

Lemma 1: The difference between a constant approximation and another approximation has at most one zero in $[-1, +1]$.

Proof: Constant approximation is $F(A,x) = a_1$ when A has the form $A=(a_1\ 0,\ a_3)$ or $A=(a_1, a_2, 0)$. Now, let non–constant another approximation function

$$F(B,x) = b_1 + b_2 \log(1 + b_3 x)$$

Due to the definition, $b_2 b_3 \neq 0$.

Consider that

$$d(x) = F(A,x) - F(B,x)$$

has two zeros in $[-1, +1]$. According to Rolle theorem

$$d'(x) = F'(A,x) - F'(B,x)$$

has zero at least for one x value. That is

$$d'(x) = - \frac{b_2 b_3}{1 + b_3 x} = 0$$

This implies $b_2 = 0$ or $b_3 = 0$. However, this is a contradiction to the assumption that $b_2 b_3 \neq 0$.

Lemma 2: The difference between a non-constant approximation and a linear approximation has at most two zeros in $[-1, +1]$.

Proof: Under the circumstances of $-1 < a_3 < +1$, consider the difference between

$F(A,x) = a_1 + a_2 \log(1 + a_3x)$ and $a_4 + a_5x$

Suppose $d(x) = F(a,x) - a_4 - a_5x$ has three zeros in $[-1, +1]$. Then derivative of $d(x)$,

$$d'(x) = \frac{a_2a_3 - a_5 - a_5a_3x}{1 + a_3x}$$

has at most zeros in $[-1, +1]$.

For the approximation function, $F(A,x) = a_1 + a_2 \log(1 + a_3x)$, to be definite in $[-1, +1]$, $1 + a_3x > 0$ is raquired. Then the right hand side of

$$(1 + a_3x)\, d'(x) = a_2a_3 - a_5 - a_5a_3x$$

is a polynomial of first degree and has at most one zero. On the other hand if d' is identically zero then

$$a_2a_3 - a_5 = 0$$

and

$$a_5a_3 = 0$$

$F(A,.)$ is another non–constant approximation, so $a_2a_3 \neq 0$. Then $a_5 = 0$. Inserting this value in the above equation we have $a_2a_3 = 0$. However, this a contradiction to the non–constant approximation, $F(A,.)$.

Lemma 3: The difference between a non–constant approximation and another approximation has at most two zeros in $[-1, +1]$.

Proof: Let $F(A,.)$ and $F(B,.)$ be two non–constant approximation functions.

Suppose $d(x) = F(A,x) - F(B,x)$ has three zeros, so $d'(x)$ has the form of

$$d'(x) = F'(A,x) - F'(B,x) = \frac{(a_2a_3 - b_2b_3) + (a_2a_3b_3 - a_3b_2b_3)x}{(1 + a_3x)\,(1 + b_3x)}$$

which must have at most two zeros. $F(A,x)$ and $F(B,x)$ to be definite in $[-1, +1]$ so that $1 + a_3x > 0$ and $1 + b_3x > 0$ are required. Then the right hand side of

$(1+a_3x)(1+b_3x)\ d'(x) = (a_2a_3-b_2b_3) + (a_2a_3b_3-a_3b_2b_3)x$
is a polinomial of the first degree so that it has at most one zero and then d has at most two zeros.

On the other hand if d' is identical to zero, d must be constant. In that case d has zeros if and only if $d'=0$. This is a contradiction. More clearly

$$a_2a_3 - b_2b_3 = 0$$

and

$$a_3b_3(a_2 - b_2) = 0$$

are required. Approximation functions are not constant, hence $a_2a_3 \neq 0$ and $b_2b_3 \neq 0$. From the second equation we find $a_2=b_2$ and inserting it in the first equation we have $a_3 = b_3$ and $d = a_1 - b_1$. Here again if d has zeros which imply $a_1 = b_1$ then we get $F(A,.) = F(B,.)$ which contradicts the assumption.

Definition 1: Define linear space D (A,.,.) formed by $\partial F(A,.) / \partial a_i$, where $i=1, 2, 3$ and let the dimention be d(A). Then d(A) evidently depends on A.

If each non-zero element of linear space D(A,.,.) has at most d(A)–1 zeros at element B of parameter space P then the space D(A,.,.) has "Clasical HAAR" property.

A linear space that has the property of clasical Haar is called Haar subspace.

Lemma 4: If D(A,.,.) correspond a constant approximation there exists a parameter A with a Haar subspace of dimention two.

Proof: Let $A=(a_1,a_2,a_3)$, then it has continious derivatives, $\partial F(A,x) / \partial a_i$:

$$\frac{\partial F(A,x)}{\partial a_1} = 1 \ ; \ \frac{\partial F(A,x)}{\partial a_2} = \log(1+a_3x) \ ; \ \frac{\partial F(A,x)}{\partial a_2} = \frac{a_2x}{1+a_3x}$$

Let $B=(b_1,b_2,b_3)$, then an element of D(A,.,.) has the following form,

$$D(A,B,x) = \sum_{i=1}^{3} b_i \frac{\partial F(A,x)}{\partial a_i} = b_1+b_2 \log(1+a_3x) + b_3 \frac{a_2x}{1+a_3x}$$

If we select the approximation function F(A,.) as contant and take $A=(a_1,0, a_3)$ then we have

$$D(A,B,x) = b_1 + b_2 \log(1+a_3x)$$

It is evidently seen that D(A,B,x) is an element of linear space of two dimentions.

On the other hand, D(A,B,x) has at most one zero in [–1, +1] according to Lemma 1, under the condition that D(A,B,x) ≠ 0.

In that case, D(A,.,.) is an "Haar subspace" of two dimetions for $A = (a_1,0, a_3)$.

Lemma 5: If F(A,.) is any non–constant approximation then D(A,.,.) is a Haar subspace of dimention 3.

Proof: Since the approximation function F(A,.) is non–constant a_2 and a_3 are non–zero and

$$D(A,B,x) = b_1 + b_2 \log(1+a_3x) + b_3 \frac{a_2x}{1+a_3x}$$

is clearly an element of vector space of dimention 3. This shows that D(A,.,.) is a linear vector space of dimention 3.

Let D(A,B,x) be a non–zero element of D (A,.,.) then $B = (b_1,b_2,b_3) \neq 0$. Since

$$D'(A,B,x) = \frac{(b_2a_3 + b_3a_2) + b_2a_3^2x}{(1 + a_3x)^2}$$

has at most one zero in [–1 , +1] then D(A,B,x) has at most two zeros. On the other hand since D'(A,B,x) = 0 then $b_2a_3 + b_3a_2 = 0$ and $b_2a_3^2 = 0$. Using $a_2 \neq 0$ and $a_3 \neq 0$ circumstances, we have $b_2=0$ and $b_3 = 0$. That is

$$D(A,B,x) = b_1$$

From the assumption $B = (b_1, b_2, b_8) \neq 0$ it is necessary to be $b_1 \neq 0$. In that case D (A,.,.) is a Haar subspace of dimention 3.

Remark 1: If A corresponds to a constant approximation function, Lemma 1 shows that d(A) =2. Otherwise Lemma 3 gives d(A) =3.

Now, to obtain a result of DE LA VALLEE–POUSSIN type which is useful in characterizing "near best approximation", let us consider a compact–Hausdorf space, X and prove some theorems.

Let us consider a compact Hausdorf space X, and a set C(X) of all continious functions on X. If P be a parameter space and f be any element of C(X) then S(A,B;x) is defined such as

$$S(A,B,x) = (F(A,x) - f(x))\ (F(A,x) - F(B,x))$$

where A and B are elements of P. Now, let us prove that

$$\rho(f) = \inf \{ \ \| F(A,.) - f \| \ ; \ A \in P \}$$

has a sublimit.

Theorem 1: Let A be an element of parameter space, P. If for each element, B, of P, there is a closed subset, K, of X such that

$$\min \{ S(A,B;x) \ ; \ x \in K \} \leqq 0$$

then

$$\rho(f) \geqq \min \{ \ |F(A,x) - f(x)| \ ; \ x \in K \} = \sigma$$

Proof: Suppose $\rho\ (f) < \sigma$ then

$$\rho(f) < \| F(B.,) - f \| < \sigma$$

such that there exists an element, B, of P. Hence for the elements x of K

$$| F(A,x) - f(x)| - |F(B,x) - f(x)| > 0$$

and

$$S(A,B,x) = | F(A,x) - f(x) |^2 - (F(A,x) - f(x))(F(B,x) - f(x))$$
$$\geq |F(A,x) - f(x)| \ (|F(A,x) - f(x)| - |F(B,x) - f(x)|) > 0$$

This contradicts the hypothesis.

Definition 2: For a g element of space C([-1, + 1]) if there exist

$$|g(x_i)| = \| g \| \ , \ g(x_i) = (-1)^i g(x_i); \ (i = 1,2,..,d(A))$$

and point set $\{ x_1, x_2, \ldots x_{d(A)+1} \}$ such that $-1 \leq x_1 < \ldots < x_{d(A)+1} \leq +1$ then g function alternates d(A) times.

Theorem 2: If approximation function F has property (Z) at A and for an element f of C([-1, + 1]), F(A,.) – f alternates on $\{x_1, x_2, \ldots x_{d(A)+1}\}$ then there exists property

$\rho(f) \geq \min\{|FA,x_k) - f(x_k)| : 1 \leq k \leq d(A)+1\}$

Proof: Since the function F(A,.) – f changes alternatively on $\{x_1, x_2, \ldots x_{d(A)+1}\}$, there exists the property

$$\mathrm{Sgn}\,(F(A,x_j) - f(x_j)) = -\,\mathrm{Sgn}\,(F(A,x_{j+1}) - f(x_{j+1})) \qquad (1)$$

where, $j = 1,2, \ldots d(A)$

Let K in theorem 1 as $K = \{x_k ; 1 \leq k \leq d(A)+1\}$ then one gets

$$\rho(f) \geq \min\{|F(A,x_k) - f(x_k)| \; ; 1 \leq k \leq d(A)+1\}$$

In that case at least for an $x_p \in K$, one gets

$$S(A,B,x_p) = (F(A,x_p) - f(x_p))\,(F(A,x_p) - F(B,x_p)) \leq 0$$

Otherwise F(A,.) – F(B,.) has d(A)+1 zeros in [–1, +1] according to the property (1). This contradicts the hypothesis that F(A,.) has property (Z) at A.

Definition 3: Approximation function F(A,.) has the property of local Haar space, with null points of degree d(A) at A, if the following conditions are fulfiled:

(I) Approxımation funtion F(A,.) has continious partial derivatives for each i, i = 1,2,.... n.

(II) Setting

$$D(A,B,x) = (B, \nabla F(A,x)) = \sum_{i=1}^{n} b_i \frac{\partial F(A,x)}{\partial a_i}$$

we have

$$F(A+B,x) - F(A,x) = D(A,B,x) + R(A,B,x)$$

and when $\|B\|$ is sufficiently small

$$R(A,B,x) = O(\|B\|)$$

(III) There exists a neighbourhood of element A which is contained in P.

(IV) Linear space D(A,.,.) is a Haar subspace of dimention d(A) in [–1, +1].

Remark 2: Approximation function F has local Haar space condition, only when D (A,.,.) obeys clasical Haar condition.

Theorem 3: If approximation function F has the local property with null points of degree d(A) at A and function f be an element of space $C([-1, +1])$, and $F(A,.)$ be the best approximation to f, then function $F(A,.) - f$ alternates $d(A)$ times.

Proof: Let F be the best approximation to f, then set of extreme points of $F(A,x)-f(x)$,

$M_A = \{x / x \in [-1, +1] : \|F(A,.) - f\| = |F(A,x) - f(x)|\}$
has at least $d(A)+1$ elements.

Under the above conditions there exist some points which hold $-1 \leq x_1 < \ldots x_{d(A)+1} < +1$ and set $\{x_1, x_2, \ldots, x_{d(A)+1}\}$ is an alternant of $F(A..) - f$. Otherwise there would be found a natural number, m, and so we can separate $[-1, +1]$ into $m+1$ subintervals such that each interval contains an extreme point and $F(A,x)-f(x)$ has same sign in these intervals.

The set of extreme points of $F(A,x)-f(x)$, has $d(A)+1$ elements, hence, for $k=1,2,\ldots d(A)$, a non-zero element B of parameter space d' can be found [2] sucht that

$$(B, \nabla F(A,x_k)) = \sum_{i=1}^{n} b_i \frac{\partial F(A,x_k)}{\partial a_i} - F(A,x_k)-f(x_k)$$

and so for all extreme points, x,

$$(F(A,x)-f(x))(B, \nabla F(A,x) = |F(A,x)-f(x)|^2$$

and then

$$\mathrm{Sgn}(B, \nabla F(A,x)) = \mathrm{Sgn}(F(A,x) - f(x))$$

This result contradicts the hypothesis of the best approximation fuction F to f.

Meinardus and Schwedt ([2] theorem 9) showed that a set M_A of extreme points, has at most $d(A)+1$ points in $[-1, +1]$.

Opposition of the Theorem 2 is corect, provided the above conditions are taken into account.

Now, combining Theorem 2 and Theorem 3 one can get the following result:

Theorem 4: If $F(A,.)$ has property (Z) at A and local Haar property with null points of degree $d(A)$, then $F(A,.)$ be the best to f if and only if $F(A,.)-f$ alternates $d(A)$ times.

Theorem 5: If F(A,.) satisfies the condition of Theorem 4, and F(A,.) is best, then it is a unique best approximation.

Proof: Suppose, F(A,.) and F(B,.) are two approximation functions. We can take $d(A) \leq d(B)$, without violating the generality.

Let set of extreme points of F(A,.) - f be $\{x_1, x_2, ..., x_{d(A)+1}\}$ $(k = 1,2, ... d(A)+1)$. According to Theorem 3, the set $\{x_1, x_3, ... x_{d(A)+1}\}$ is an alternant of F(A..)-f. Then we have

$$F(A,x_{j+1}) - f(x_{j+1}) = -(F(A,x_j) - f(x_j))$$

where, $j = 1,2,..., d(A)$. Hence using Equation 1 we get inequalities system

$$\begin{aligned} F(A,x_1) - F(B,x_1) &\leq 0 \\ F(A,x_2) - F(B,x_2) &\geq 0 \\ &\dots\dots \end{aligned}$$

or

$$\begin{aligned} F(A,x_1) - F(B,x_1) &\geq 0 \\ F(A,x_2) - F(B,x_2) &\leq 0 \\ &\dots\dots \end{aligned}$$

It is sufficient to investigate the first part,

$$\begin{aligned} F(A,x_1) - F(B,x_1) &\leq 0 \\ F(A,x_2) - F(B,x_2) &\geq 0 \\ &\dots\dots \end{aligned}$$

If the inequalities had been certain, F(A,.) - F(B,.) would have had $d(A)+1$ definite null points and from the Haar condition we would have gotten result

$$F(,.) = F(B,.)$$

On the other hand, if the inequalities had been correct for a k_o, we would have gotten

$$F(A,x_{ko}) - F(B,x_{ko}) \neq 0$$

$$\text{Sng}\,(F(A,x_{ko}) - F(B,x_{ko})) = (-1)^{ko}$$

However, if (F(,.) and F(B,.) are two approximation functions and if we take

$$\begin{aligned} A(t) &= (1-t)\,A + t\,B \\ B(t) &= (1-t)\,B + t\,A \end{aligned}$$

then F(A(t),.) and F(B(t),.) are also approximation functions. If ve denote $\delta = B - A$ in

$$B(t) = B - t\ (B - A)$$

we get

$$B(t) = B - t\ \delta$$

where, parameter δ is an element of space p.

Since D(B,.,.) satisfies Haar condition, each non-zero element of D(B,.,.) has at most d(A)–1 null points at element δ of parameter space P. So F(B,.) have local Haar property.

Using property (II) of local Haar condition in F(B,x)–F(B–tδ,x) we get

$$F(B,x) - F(B-t\delta,x) = tD(B,\delta,x) + R(B,\delta,x)$$

and adding the approximation function F(A..) to the each side of this equation and denoting R(B,δ, x) = 0 (t), we find

$$F(A,x) - F(B-t\delta,x) = F(A,x) - F(B,x) + tD(B,\delta,x) + 0(t)$$

We get the following system, for $t > 0$,

$$\begin{aligned} F(A,x_1) - F(B-t\delta,x_2) &< 0 \\ F(A,x_2) - F(B-t\delta,x_2) &> 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Thus F(A,.) – F(B–tδ,.) has at least d(A) null points in [–1, +1] and when t is approaching to zero we get

$$F(A,.) = F(,.)$$

ÖZET

Chebyshev yaklaşımı üzerine daha önce yapılan çalışmalar aydınlatılmış, [0,α] üzerine en iyi Chebyshev yaklaşımının A+B* log (1+CX) olduğu ispatlanmış ve yeni kavramlar yardımıyla konu genelleştirilmiştir.

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