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RELATIVE SUBCOPURE-INJECTIVE MODULES

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Abstract. In this paper, copure-injective modules are examined from an alternative perspective. For two modules A and B, A is called B-subcopure-injective if for every copure monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. The class $\underline{\mathfrak{CPI}}^{-1}(A) = \{B : A \text{ is } B\text{-subcopure-injective}\}$ is called the subcopure-injectivity domain of A. We obtain characterizations of copure-injective modules, right CDS rings and right V-rings with the help of subcopure-injectivity domains. Since subcopure-injectivity domains clearly contains all copure-injective modules, studying the notion of modules which are subcopure-injective only with respect to the class of copure-injective modules is reasonable. We refer to these modules as sc-indigent. We studied the properties of subcopure-injectivity domains and of sc-indigent modules and investigated these modules over some certain rings.

1. Introduction and preliminaries

Throughout this paper, R will denote an associative ring with identity, and modules will be unital right R-modules, unless otherwise stated. As usual, the category of right R-modules is denoted by $Mod-R$.

Some new studies in module theory have focused on to approach to the injectivity from the point of relative notions. The injectivity domain $\mathfrak{In}^{-1}(A)$ for a module A, is the class of all modules B such that A is B-injective [1]. Given A and B modules, A is called B-subinjective if for every monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. Instead of using the injectivity domain, in latest articles, authors have proposed to consider an alternative sight so-called subinjectivity domain $\underline{\mathfrak{In}}^{-1}(A)$, contains of modules B such that A is B-subinjective ([2]). It is clear that injectivity of A is equivalent to that $\underline{\mathfrak{In}}^{-1}(A) = Mod-R$. If B is injective, then A is exactly B-subinjective. So by [2, Proposition 2.3], the class of injective modules is the smallest

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possible subinjectivity domain. The recent studies of non-injective modules have been made to figure out the notion of modules that are subinjective only with respect to the class of injective modules. This kind of non-injective modules are called indigent in [2]. So far, it is not known whether the existence of indigent modules for an arbitrary ring, but a positive answer is known for some rings, such as Noetherian rings ([3, Proposition 3.4]).

A submodule A of a right R-module B is said to be pure if for every left R-module K the natural induced map $i \otimes 1_K : A \otimes K \to B \otimes K$ is a monomorphism. Recall that a module A is said to be B-pure-injective if for every pure monomorphism $f : C \to B$ and every homomorphism $g : C \to A$, there exists a homomorphism $h : B \to A$ such that $hf = g$. A module A is said to be pure-injective if it is B-pure-injective for every module B. As an analogue to the injectivity profile of [12], the pure-injectivity profile of a ring is introduced in [5]. The pure-injectivity domain $\mathfrak{PI}^{-1}(A)$ of a module A, consists of those modules B such that A is B-pure-injective. Inspired by the notion of subinjectivity, the notion of pure-subinjectivity introduced in [11]. A module A is called B-pure-subinjective if for every pure monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. The pure-subinjectivity domain of a module A is the class $\underline{\mathfrak{PI}}^{-1}(A) = \{B : A \text{ is } B\text{-pure-subinjective}\}$. If B is pure-injective, then A is exactly B-pure-subinjective. So by [11, Theorem 2.4], for a module A, the class $\underline{\mathfrak{PI}}^{-1}(A)$ must contain the class of pure-injective modules at least. In [11], modules whose pure-subinjectivity domain consists of only pure-injective modules is called pure-subinjectively poor (ps-poor for short).

An R-module A is said to be finitely embedded (or cofinitely generated) if $E(A) = E(S_1) \oplus E(S_2) \oplus ... \oplus E(S_n)$, where $S_1, S_2, ..., S_n$ are simple R-modules (see [16]). If an R-module A is isomorphic to $\prod\{E(S_\alpha) | S_\alpha \text{is a simple right } R\text{-module}, \alpha \in I\}$, where I is some index set, then A is called a cofree module (see [6]). A right R-module A is said to be cofinitely related if there is an exact sequence $0 \to A \to B \to C \to 0$ of R-modules with B finitely embedded, cofree and C finitely embedded (see [6]). As a dual notion of purity, by using cofinitely related modules, the notion of copurity is introduced in [7]. An exact sequence of R-modules $0 \to A \to B \to C \to 0$ is called a copure exact sequence if every cofinitely related right R-module is injective relative to this sequence.

Following idea on pure-injectivity profile of [5], in [15], the copure-injectivity profile of a ring is introduced. For two modules A and B, A is called B-copure-injective if for every copure monomorphism $f : C \to B$ and a homomorphism $g : C \to A$, there exists a homomorphism $h : B \to A$ such that $hf = g$. A is copure-injective if it is injective with respect to every copure exact sequences (see [8]). The copure-injectivity domain $\mathfrak{CPI}^{-1}(A)$ of A is the class of modules B such that A is B-copure-injective. In [15], copure-injectively-poor (shortly copi-poor) modules introduced as modules with minimal copure-injectivity domain and studied properties of copi-poor modules. The existence of copi-poor modules are studied and investigated over some certain rings, but we do not know whether copi-poor modules exist over arbitrary rings (see [15]).

Inspired by the notion of pure-subinjectivity from [11], in this paper we initiate the study of an alternative perspective on the analysis of the copure-injectivity of a module, as we introduce the notions of relative subcopure-injectivity and assign to every module its subcopure-injectivity domain. The aim of this paper is to investigate the viability of obtaining valuable information about a ring R from the perspective of subcopure-injectivity domain.

In Section 2, relative subcopure-injectivity and subcopure-injectivity domains of modules introduced. We investigate the properties of the notion of subcopure-injectivity and we compare subcopure-injectivity domains with (copure-)injectivity domains. We obtain characterizations of copure-injective modules, right CDS rings and right V-rings with the help of subcopure-injectivity domains.

In section 3, we introduced and studied the concept of cc-injective modules in terms of relative subcopure-injective modules. We give examples of cc-injective modules and compare cc-injective modules with cotorsion modules in Example 19. We prove that R is a right V-ring if and only if every cc-injective right R-module is injective. We investigate when the class of B-subcopure-injective modules is closed under extensions.

An R-module is copure-injective if and only if its subcopure-injectivity domain consists of $Mod-R$. Since subcopure-injectivity domains clearly contain all copure-injective modules, it is reasonable to investigate modules which are subcopure-injective only with respect to the class of copure-injective modules. It is thus to keep in line with [11], we refer to these modules as sc-indigent. In Section 4 of this paper, we studied and investigated sc-indigent modules over some certain rings. We compared sc-indigent modules with indigent modules and ps-poor modules.

2. Relative subcopure-injective modules

In this section, we study the B-subcopure-injective modules for a module B and examine its fundamental properties.

Definition 1. *For two modules A and B, A is called B-subcopure-injective if for every copure monomorphism $f : B \to C$ and homomorphism $g : B \to A$, there exists a homomorphism $h : C \to A$ such that $hf = g$. The class $\underline{\mathfrak{CPI}}^{-1}(A) = \{B : A \text{ is } B\text{-subcopure-injective}\}$ is called the subcopure-injectivity domain of A.*

Hiremath proved in [8, Theorem 7] that every module can be embedded as a copure submodule in a direct product of cofinitely related modules. By [8, Proposition 3], every cofinitely related module is copure-injective and every direct product of copure-injective modules is copure-injective. This gives the below result that we use frequently in the sequel.

Lemma 2. *For every module A, there exists a copure monomorphism $\alpha : A \to C$ with C is copure-injective.*

Our next Lemma gives a characterization of the B-subcopure-injective modules for a module B.

Lemma 3. *Let A and B be two modules. The following conditions are equivalent:*

(1) *A is B-subcopure-injective.*
(2) *For every homomorphism $g : B \to A$ and every copure monomorphism $\alpha : B \to C$ with C copure-injective, there exists $h : C \to A$ such that $h\alpha = g$.*
(3) *For every homomorphism $g : B \to A$ and every copure monomorphism $\alpha : B \to C$ with C direct product of cofinitely related modules, there exists $h : C \to A$ such that $h\alpha = g$.*
(4) *For every $g : B \to A$ there exist a copure monomorphism $\alpha : B \to C$ with C copure-injective and $h : C \to A$ such that $h\alpha = g$.*

Proof. (1) $\Rightarrow$ (2) Obvious. (2) $\Rightarrow$ (3) It follows from [8, Proposition 3].

(3) $\Rightarrow$ (4) Let $g : B \to A$ be a homomorphism. By Lemma 2, there exists a copure monomorphism $\alpha : B \to C$ with C copure-injective, whence C is a direct summand of F where $F = \prod_{i \in I} F_i$ with each F_i cofinitely related by [8, Theorem 8]. So $i\alpha : B \to F$ is copure monomorphism where $i : C \to F$. By (3), there exists $h : F \to A$ such that $(hi)\alpha = h(i\alpha) = g$,where $i\alpha : B \to F$.

(4) $\Rightarrow$ (1) Let $g : B \to A$ be a homomorphism and $\bar{\alpha} : B \to D$ a copure monomorphism. By (4), there exists a monic copure map $\alpha : B \to C$ with C copure-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$. So by the copure-injectivity of C, there exists a homomorphism $\bar{h} : D \to C$ such that $\alpha = \bar{h}\bar{\alpha}$. Then $h\bar{h} : D \to A$ and $h\bar{h}\bar{\alpha} = h\alpha = g$. Hence, A is B-subcopure-injective. □

Proposition 4. *Let A be an R-module. The following conditions are equivalent:*

(1) *A is copure-injective.*
(2) $\underline{\mathfrak{CPI}}^{-1}(A) = Mod - R$.
(3) *A is A-subcopure-injective.*

Proof. (1) $\Rightarrow$ (2) For any R-module B and any copure-injective module A, every copure monomorphism $\alpha : B \to D$ and a homomorphism $g : B \to A$, there exists a homomorphism $h : D \to A$ such that $h\alpha = g$. Hence, A is B-subcopure-injective and so $B \in \underline{\mathfrak{CPI}}^{-1}(A)$. Consequently, $\underline{\mathfrak{CPI}}^{-1}(A) = Mod - R$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Assume that A is A-subcopure-injective. For any copure monomorphism $\alpha : A \to B$ with B copure-injective and $1_A : A \to A$, there exists a homomorphism $g : B \to A$ such that $g\alpha = 1_A$. Thus α splits. This means that A is copure-injective. □

The next result asserts that subcopure-injectivity domain $\underline{\mathfrak{CPI}}^{-1}(A)$ of A how small can be. It should contain the copure-injective modules at least.

Proposition 5. $\bigcap_{A \in Mod-R} \underline{\mathfrak{CPI}}^{-1}(A) = \{C \in Mod - R \mid C$ *is copure-injective*$\}$.

Proof. Suppose that each R-module is B-subcopure-injective for an R-module B. Then, by Proposition 4, B is copure-injective. Conversely, let A be any R-module and B a copure-injective module. Let $g : B \to A$ be a homomorphism and $\alpha : B \to C$ a copure monomorphism. Since B is copure-injective, the splitting map $\alpha : B \to C$ gives the homomorphism $\beta : C \to B$ such that $\beta\alpha = 1_B$. So $\beta(\alpha g) = (\beta\alpha)g = g$. Hence $B \in \underline{\mathfrak{CPI}}^{-1}(A)$ for any R-module A. □

Clearly, $\underline{\mathfrak{CPI}}^{-1}(A)$ contains $\underline{\mathfrak{In}}^{-1}(A)$ for any module A. The following example shows that equality need not hold.

Example 6. *Let $G = Z(n)$ be a cyclic group of order n. Since G is finite it is cofinitely related and so it is copure-injective $\mathbb{Z}$-module* [8, Proposition 3]. *So $G \in \underline{\mathfrak{CPI}}^{-1}(G)$ by Proposition 4. But $G \notin \underline{\mathfrak{In}}^{-1}(G)$, otherwise G would be an injective $\mathbb{Z}$-module.*

It is natural to investigate conditions to get the coincidence of the injectivity, and subcopure-injectivity domains, either for a certain class of modules or all the modules in $Mod-R$. We start by proving that, for all modules, subcopure-injectivity domains are the same as their subinjectivity domains over a right V-ring. Recall that a ring R is a right V-ring if and only if all exact sequences in $Mod-R$ are copure if and only if all copure-injective modules are injective (see [8, Proposition 5]).

Corollary 7. *Let R be a ring. The following conditions are equivalent:*

(1) *R is a right V-ring.*
(2) *$\underline{\mathfrak{CPI}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}(A)$ for each R-module A.*
(3) *$\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \underline{\mathfrak{In}}^{-1}(A)$ for each R-module A.*

Proof. (1) $\Rightarrow$ (2) It is easy since for any module A, over a right V-ring its extension is copure.
(2) $\Rightarrow$ (3) It is obvious.
(3) $\Rightarrow$ (1) For a copure injective right R-module A, by Proposition 4, $A \in \underline{\mathfrak{CPI}}^{-1}(A)$. By (3), $A \in \underline{\mathfrak{In}}^{-1}(A)$. This says that A is injective, and so R is a right V-ring by [8, Proposition 5].

□

Proposition 8. *Let A be a module. The following conditions are equivalent:*

(1) *A is copure-injective.*
(2) *$\underline{\mathfrak{CPI}}^{-1}(A)$ is closed under copure submodules.*
(3) *$\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{CPI}^{-1}(A)$.*
(4) *$\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{CPI}^{-1}(A)$.*

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear since $\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{CPI}^{-1}(A) = Mod-R$.

(2) $\Rightarrow$ (1) For a copure-injective extension C of A, $C \in \underline{\mathfrak{CPI}}^{-1}(A)$, so A is also in $\underline{\mathfrak{CPI}}^{-1}(A)$ by (2). Then by Proposition 4, A is copure-injective.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) For a copure-injective extension C of A, $C \in \underline{\mathfrak{CPI}}^{-1}(A)$. This implies that A is C-copure-injective i.e. $C = A \oplus B$ for some submodule B of A, whence A is copure-injective. □

The rings for which every right R-module is copure-injective are called right CDS, [8, Corollary 18]. As a result of Proposition 8, we get the following Corollary.

Corollary 9. *Let R be a ring. The following conditions are equivalent:*

(1) *R is right CDS.*
(2) $\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{CPI}^{-1}(A)$ *for each R-module A.*
(3) $\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{CPI}^{-1}(A)$ *for each R-module A.*

Proof. (2) $\Rightarrow$ (3) It is clear.

(1) $\Rightarrow$ (2) Let A be an R-module. Since R is a right CDS ring, A is copure-injective. The rest follows from Proposition 8.

(3) $\Rightarrow$ (1) For any right R-module A, $\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{CPI}^{-1}(A)$ by the hypothesis. Thus every right R-module A is copure-injective by Proposition 8, whence R is right CDS. □

Remark 10. *If A is R-subcopure-injective, for a ring R and a module A, then $\underline{\mathfrak{CPI}}^{-1}(A)$ and $Mod-R$ need not be equal. For example if R is copure-injective ring that is not CDS, then for every module A, A is R-subcopure-injective by Proposition 5. But by the definition of right CDS ring, we can find a module A that is not copure-injective.*

Proposition 11. *Let A be a module. The following conditions are equivalent:*

(1) *A is injective.*
(2) $\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{In}^{-1}(A)$.
(3) $\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) By the copure-injectivity of $E(A)$, $E(A) \in \underline{\mathfrak{CPI}}^{-1}(A)$. By (3), $E(A) \in \mathfrak{In}^{-1}(A)$, and hence A is injective. □

Corollary 12. *Let R be a ring. The following conditions are equivalent:*

(1) *R is semisimple.*
(2) $\underline{\mathfrak{CPI}}^{-1}(A) = \mathfrak{In}^{-1}(A)$ *for each R-module A.*
(3) $\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$ *for each R-module A.*

Proof. (2) $\Rightarrow$ (3) It is clear.

(1) $\Rightarrow$ (2) Let A be an R-module. Since R is semisimple, A is injective. The rest follows from Proposition 11.

(3) ⇒ (1) For any right R-module A, $\underline{\mathfrak{CPI}}^{-1}(A) \subseteq \mathfrak{In}^{-1}(A)$ by the hypothesis. Thus every right R-module A is injective by Proposition 11, whence R is semisimple. □

In general, factors of copure-injective modules need not be copure-injective (see, [8, Remark 24]). But if R is a Dedekind domain, every copure factor of copure-injective module is copure-injective by [8, Corollary 28]. Hence, by the following Proposition, $\underline{\mathfrak{CPI}}^{-1}(A)$ is closed under copure homomorphic images over Dedekind domains for a module A.

Proposition 13. *$\underline{\mathfrak{CPI}}^{-1}(A)$ is closed under copure quotients for any module A if and only if every copure homomorphic image of a copure-injective module is copure-injective.*

Proof. Let B be a copure submodule of copure-injective module A. Since $A \in \underline{\mathfrak{CPI}}^{-1}(\frac{A}{B})$, by the hypothesis $\frac{A}{B} \in \underline{\mathfrak{CPI}}^{-1}(\frac{A}{B})$, and so $\frac{A}{B}$ is copure-injective. Conversely, let A be a module and C a copure submodule of B with $B \in \underline{\mathfrak{CPI}}^{-1}(A)$. By Lemma 2, there exists a copure monomorphism $\alpha : B \to D$ with D copure-injective. Let $f : \frac{B}{C} \to A$ be any homomorphism. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\alpha} & D & \longrightarrow & \frac{D}{B} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \| & & \\
0 & \longrightarrow & \frac{B}{C} & \xrightarrow{\alpha'} & E & \longrightarrow & \frac{D}{B} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & & & & & \\
 & & A & & & & & &
\end{array}$$

where $\pi : B \to \frac{B}{C}$ is the natural epimorphism. By commutativity of the following diagram:

$$\begin{array}{ccc}
B & \xrightarrow{\alpha} & D \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi''} \\
\frac{B}{C} & \xrightarrow{\alpha''} & \frac{D}{C}
\end{array}$$

and the pushout diagram property, there exists a map $\phi : E \to \frac{D}{C}$ such that $\phi\pi' = \pi''$ and $\phi\alpha' = \alpha''$. Since A is B-subcopure-injective, there exists a homomorphism $\varphi : D \to A$ such that $\varphi\alpha = f\pi$. Then, $\varphi(C) = \varphi\alpha(C) = f\pi(C) = f(0) = 0$. Hence, $Ker(\phi\pi') \subseteq Ker\varphi$, and so there exists $\psi : \frac{D}{C} \to A$ such that $\psi\pi'' = \varphi$. For every $x \in B$, $\psi(x + C) = \psi\pi''(x) = \varphi(x) = f\pi(x) = f(x + C)$. Thus ψ extends f. Then by the hypothesis, $\frac{D}{C}$ is copure-injective, so by Lemma 3, $\frac{B}{C} \in \underline{\mathfrak{CPI}}^{-1}(A)$. □

Proposition 14. $\underline{\mathfrak{CPI}}^{-1}(\prod_{i\in I} A_i) = \bigcap_{i\in I} \underline{\mathfrak{CPI}}^{-1}(A_i)$ *for any set of modules* $\{A_i\}_{i\in I}$.

Proof. Let $B \in \underline{\mathfrak{CPI}}^{-1}(\prod_{i\in I} A_i)$, $i \in I$ and $f : B \to A_i$ be a homomorphism. Then there exists a homomorphism $g : C \to \prod_{i\in I} A_i$ such that $g\alpha = i_{A_i} f$, where $\alpha : B \to C$ is the monic map with C copure-injective and $i_{A_i} : A_i \to \prod_{i\in I} A_i$ is the inclusion map. Let $\pi_{A_i} : \prod_{i\in I} A_i \to A_i$ denote the natural projection. Since $\pi_{A_i} g\alpha = \pi_{A_i} i_{A_i} f = f$, f is extended to $\pi_{A_i} g$. Therefore $B \in \underline{\mathfrak{CPI}}^{-1}(A_i)$ for any $i \in I$. Conversely , let $B \in \underline{\mathfrak{CPI}}^{-1}(A_i)$ for all $i \in I$ and $f : B \to \prod_{i\in I} A_i$. Hence for each $i \in I$, there exists $g_i : C \to A_i$ with $g_i\alpha = \pi_{A_i} f$. Now define $g : C \to \prod_{i\in I} A_i$ by $x \mapsto g_i(x)$. Since $g\alpha = f$, g extends f. Thus, $B \in \underline{\mathfrak{CPI}}^{-1}(\prod_{i\in I} A_i)$. □

Corollary 15. *Let B be a module. Then B-subcopure-injective modules are closed under direct summands and finite direct sums.*

Proof. Let A be a module with decomposition $A = \oplus_{i=1}^{n} A_i$. By Proposition 14, $B \in \underline{\mathfrak{CPI}}^{-1}(A)$ if and only if $B \in \bigcap_{i=1}^{n} \underline{\mathfrak{CPI}}^{-1}(A_i)$. Now the result follows. □

The following shows that Proposition 14 do not hold for infinite direct sums.

Example 16. *Let $K_i = \mathbb{Z}_{p_i}$ and $G = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{p_i}$ where p_i is a prime integer for all $i \in \mathbb{N}$. Since every $\mathbb{Z}_{p_i}$ is pure-injective, every $\mathbb{Z}_{p_i}$ is copure-injective by* [8, Proposition 9]*. So $G \in \underline{\mathfrak{CPI}}^{-1}(\mathbb{Z}_{p_i})$ for all $i \in \mathbb{N}$. But $G \notin \underline{\mathfrak{CPI}}^{-1}(G)$ since G is not copure-injective by* [8, Examples-(ii)].

Proposition 17. *If $B \in \underline{\mathfrak{CPI}}^{-1}(A)$, then every direct summand of B is in $\underline{\mathfrak{CPI}}^{-1}(A)$.*

Proof. Suppose C is a direct summand of B, and let $f : C \to A$ be a homomorphism. By Lemma 2, there exist copure monomorphisms $i : B \to D$ and $j : C \to E$ with D and E copure-injective. Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & C & \xrightarrow{i_C} & B \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ & & E & & D \end{array}$$

where $i_C : C \to B$ the inclusion map. Since D is copure-injective, there exists $h : E \to D$ such that $hj = ii_C$. Let $\pi_C : B \to C$ be the projection map. Since A is B-subcopure-injective, there exists a homomorphism $g : D \to A$ such that $gi = f\pi_C$. Then, $(gh)j = g(hj) = gii_C = f\pi_C i_C = f$, and so by Lemma 3, A is C-subcopure-injective. □

3. CC-INJECTIVE MODULES

In this section, we introduced and studied the concept of cc-injective modules in terms of relative subcopure-injective modules.

A module C is said to be co-absolutely co-pure (c.c. in short) if every exact sequence of modules ending with C is copure, equivalently $Ext^1_R(C, A) = 0$ for every co-finitely related module A. Clearly every projective module is c.c. But the converse need not be true, for instance, the additive group $\mathbb{Q}$ is a c.c. $\mathbb{Z}$-module but $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module (see, [9, Example on page 290]).

Definition 18. *A right module A is called cc-injective if $Ext^1_R(B, A) = 0$ for any c.c. module B.*

Recall that a module A is called cotorsion if $Ext^1_R(B, A) = 0$ for every flat module B. A module A is called linearly compact if any family of cosets having the finite intersection property has a nonempty intersection. A commutative ring is called classical if the injective hull $E(S)$ of all simple modules S are linearly compact (see [17, §3]).

Example 19. *(1) By definition, any cofinitely related module is cc-injective.*
(2) By [9, Remark 15]*, c.c. modules need not be flat in general. By* [9, Corollary 14] *c.c. modules are flat over a commutative ring. So, in this case every cotorsion module is cc-injective.*
(3) By [9, Remark 12]*, flat modules need not be c.c. Over a commutative classical ring flat modules are c.c. by* [9, Proposition 11]*. So, in this case every cc-injective module is cotorsion.*

Remark 20. *Over a commutative ring R every simple R-module is cotorsion by* [13, Lemma 2.14]*. So by Example 19(2), every simple R-module is cc-injective.*

Lemma 21. *Every copure-injective module is cc-injective.*

Proof. Let A be a copure-injective module and B a c.c. module. By [9, Proposition 5], there exists a copure exact sequence $0 \to D \to P \to B \to 0$ with P projective. If we apply $Hom(-, A)$ to this sequence, we have $Hom(P, A) \to Hom(D, A) \to Ext^1_R(B, A) \to Ext^1_R(P, A) = 0$. Since A is copure-injective, $Hom(P, A) \to Hom(D, A)$ is epic, and so $Ext^1_R(B, A) = 0$ for any c.c. module B. Hence A is cc-injective. □

Proposition 22. *For a ring R, the following conditions are equivalent:*

(1) *R is a right V-ring.*
(2) *Every copure-injective right R-module is injective.*
(3) *Every cc-injective right R-module is injective.*

Proof. (1) ⇔ (2) It follows by [8, Proposition 5].

(3) ⇒ (2) It immediately from Lemma 21.

(1) $\Rightarrow$ (3) Let A be a cc-injective R-module and B any R-module. Since R is right V, B is a c.c. module by [9, Proposition 4]. Thus $Ext^1_R(B, A) = 0$ for any R-module B, and so A is injective. □

Proposition 23. *Let B be an R-module and $\alpha : B \to C$ a copure monomorphism with C copure-injective. If $C/im(\alpha)$ is c.c., then every cc-injective module is B-subcopure-injective.*

Proof. Let A be a cc-injective module and $C/im(\alpha)$ a c.c. module. Applying functor $Hom(-, A)$ to the exact sequence $0 \to B \to C \to C/im(\alpha) \to 0$, we have $Hom(C, A) \to Hom(B, A) \to Ext^1_R(C/im(\alpha), A)$. Since $C/im(\alpha)$ is c.c., $Ext^1_R(C/im(\alpha), A) = 0$ and so $Hom(C, A) \to Hom(B, A)$ is epic. Hence A is B-subcopure-injective by Lemma 3. □

Theorem 24. *Let A and B be two modules. Consider the following conditions:*

(1) *A is B-subcopure-injective.*
(2) *For every homomorphism $g : B \to A$, there exist a monomorphism $\alpha : B \to C$ with C copure-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$.*
(3) *For every homomorphism $g : B \to A$, there exist a monomorphism $\alpha : B \to C$ with C cc-injective and a homomorphism $h : C \to A$ such that $h\alpha = g$.*
(4) *For every homomorphism $g : B \to A$ and for any extension $\alpha : B \hookrightarrow C$ with C/B is c.c., there exists $h : C \to A$ such that $h\alpha = g$.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *Also, if $D/im(\alpha)$ is c.c. for a copure monomorphism $\alpha : B \to D$ with D copure-injective, then* (4) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Obvious by Lemma 3.

(2) $\Rightarrow$ (3) It follows from Lemma 21, since every copure-injective module is cc-injective.

(2) $\Rightarrow$ (1) Let $\alpha : B \to C$ be a copure-monomorphism and $g : B \to A$ a homomorphism. By (2), exists a monomorphism $\beta : B \to D$ with D copure-injective and a homomorphism $h : D \to A$ such that $h\beta = g$. Since D is copure-injective, there exists a homomorphism $f : C \to D$ such that $f\alpha = \beta$. Hence, $(hf)\alpha = h\beta = g$, and so (1) follows.

(3) $\Rightarrow$ (4) Let C be an extension of B with C/B is c.c. and $g : B \to A$ a homomorphism. So, $0 \to B \xrightarrow{\alpha} C \to C/B \to 0$ is copure exact. Then consider the exact sequence with E cc-injective:

$0 \to Hom_R(C/B, E) \to Hom_R(C, E) \xrightarrow{\alpha^*} Hom_R(B, E) \to Ext^1_R(C/B, E) = 0$

Since, α^* is surjective, by (3), there exists a monomorphism $f : B \to E$ and a homomorphism $h : E \to A$ such that $hf = g$. Since α^* is surjective, there exists a homomorphism $\beta : C \to E$ such that $\beta\alpha = f$. Hence, $h(\beta\alpha) = hf = g$, and so (4) follows.

$(4) \Rightarrow (1)$: Let $\alpha : B \to D$ be a copure monomorphism with D copure-injective and $D/im(\alpha)$ is c.c. So, by (4), for any homomorphism $g : B \to A$ there exists $h : D \to A$ such that $h\alpha = g$. Thus A is B-subcopure-injective by Lemma 3. □

Now we investigate when the class of B-subcopure-injective modules is closed under extensions.

Proposition 25. *Let B be an R-module and $\alpha : B \to C$ a copure monomorphism with C copure-injective. The class of B-subcopure-injective modules is closed under extensions if and only if for every exact sequence $0 \to A' \to A \to C \to 0$ with A' B-subcopure-injective, A is B-subcopure-injective.*

Proof. Let $0 \to A' \to A \to C \to 0$ be an exact sequence with A' B-subcopure-injective. Since C is copure-injective, it is B-subcopure-injective. By the hypothesis, A is B-subcopure-injective. Conversely, let $0 \to A' \to A \xrightarrow{\pi} A'' \to 0$ be an exact sequence with A' and A'' B-subcopure-injective. Then by Lemma 3, for every map $g : B \to A$, there exists a map $h : C \to A''$ such that $\pi g = h\alpha$ where $\alpha : B \to C$ is the copure monomorphism with C copure-injective. If we consider the pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & D & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow h & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \xrightarrow{\pi} & A'' & \longrightarrow & 0 \end{array}$$

there exists a homomorphism $\gamma : B \to D$ such that $f\gamma = g$ and $\beta\gamma = \alpha$. By hypothesis, D is B-subcopure-injective, so by Lemma 3, there exists a homomorphism $h' : C \to D$ such that $h'\alpha = \gamma$. Thus, $fh'\alpha = f\gamma = g$ and so, A is B-subcopure-injective by Lemma 3. □

A ring R is said to be right co-noetherian if every homomorphic image of a finitely embedded R-module is finitely embedded, equivalently for each simple right R-module S the injective hull $E(S)$ is Artinian (see [10, Theorem]). Over a commutative noetherian ring, the injective hull of each simple right R-module is Artinian by [14, Exercise 4.17]. Thus every commutative Noetherian ring is co-noetherian. In the following, for an ideal I, we deal with an R-module structure of an R/I-module.

Proposition 26. *Let R be a right co-noetherian ring and $f : R \to S$ a ring epimorphism. If A is cc-injective S-module, then A is cc-injective R-module.*

Proof. Let A be a cc-injective S-module. Since $f : R \to S$ is a ring epimorphism, $S \cong R/I$ for some ideal I of R and so A can be considered as R/I-module. Let C be an extension of A by a c.c. module F as R-modules. Since F is c.c., the exact sequence $0 \to A \to C \to F \to 0$ is copure. Then $A \cap CI = AI$ for each right ideal I by [7, proposition 16]. Since A is an R/I-module, $A \cap CI = AI = 0$, and so $\frac{A+CI}{CI} \cong A$. Thus we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{A+CI}{CI} & \longrightarrow & \frac{C}{CI} & \longrightarrow & \frac{C}{A+CI} & \longrightarrow & 0 \end{array}$$

Since $\frac{C}{A} \otimes \frac{R}{I} \cong \frac{C}{A+CI}$ is c.c. as an R/I-module, so the second exact sequence splits and so does the first. Hence $Ext^1_R(F, A) = 0$, and A is cc-injective R-module. □

4. SC-INDIGENT MODULES

Indigent (resp. ps-poor) modules were introduced and some results about them were obtained in [2] (resp. [11]). Proposition 5 says that subcopure-injectivity domain of any module A contains all copure-injective modules, so studying the notion of modules which are subcopure-injective only with respect to the class of copure-injective modules is reasonable. It is thus to keep in line with [2], we refer to these modules as subcopure-injectively indigent (sc-indigent for short). In this section, sc-indigent modules investigated over certain rings and compared these modules with indigent modules and ps-poor modules.

Definition 27. *A module A is said to be subcopure-injectively indigent (sc-indigent for short), if $\underline{\mathfrak{CPI}}^{-1}(A)$ consists of only copure-injective modules.*

Remark 28. *Let A be a module with decomposition $A = B \oplus C$. If B is sc-indigent, then so is A, by Proposition 14.*

Proposition 29. *For a ring R, the following conditions are equivalent:*

(1) *R is right CDS.*
(2) *Every R-module is sc-indigent.*
(3) *There exists a copure-injective sc-indigent R-module.*
(4) 0 *is an sc-indigent R-module.*
(5) *R has an sc-indigent module and every sc-indigent R-module is copure-injective.*
(6) *R has an sc-indigent module and every factor of an sc-indigent R-module is sc-indigent.*
(7) *R has an sc-indigent module and every summand of an sc-indigent R-module is sc-indigent.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (5) are clear since every R-module is copure-injective.
The implications (2) ⇒ (4) and (2) ⇒ (6) ⇒ (7) are clear.
(4) ⇒ (2) It immediately from Remark 28.
(2) ⇒ (3) The copure-injective extension C of any module A is sc-indigent.
(3) ⇒ (1) Let C be a copure-injective sc-indigent module and A a module. Since C is A-subcopure-injective, A is copure-injective. Then R is a right CDS ring.

(5) ⇒ (1) By (5), there exist an sc-indigent module B. Then $A \oplus B$ is also sc-indigent for any module A by Remark 28. So A is copure-injective by (5). Also A is copure-injective. Thus R is a right CDS ring.
(7) ⇒ (2) Let A be an R-module. Then $A \oplus B$ is an sc-indigent module for some sc-indigent module B. Hence, A is sc-indigent by the hypothesis. □

Remark 30. *Over a commutative uniserial ring R, every R-module is sc-indigent since such rings are CDS by* [4, Theorem 10.4].

Remark 31. *An sc-indigent module need not be indigent. Consider the ring $R = \mathbb{Z}/p^2\mathbb{Z}$, for some prime integer p. R is an artinian principal ideal ring. Hence it is a CDS-ring by* [4, Theorem 10.4]*. So every R-module is sc-indigent. Since $\mathbb{Z}/p^2\mathbb{Z}$ is injective $\mathbb{Z}/p^2\mathbb{Z}$-module, $\underline{\mathfrak{In}}^{-1}(\mathbb{Z}/p^2\mathbb{Z}) = Mod-R$. But since R is not a semisimple ring, $\mathbb{Z}/p^2\mathbb{Z}$ is not an indigent R-module.*

Remark 32. *An indigent module need not be sc-indigent. Let R be a commutative Noetherian ring which is not CDS and Γ a complete set of representatives of finitely presented right R-modules. Set $F := \bigoplus_{S_i \in \Gamma} S_i$. Thus the character module F^+ of F is a pure-injective indigent R-module by* [3, Proposition 3.4]*. Since R is commutative, F^+ is copure-injective by* [8, Proposition 9]*, and so $\underline{\mathfrak{CPI}}^{-1}(F^+) = Mod - R$. But since R is not a CDS-ring, F^+ is not an sc-indigent R-module.*

Proposition 33. *Indigent modules and sc-indigent modules coincide over a right V-ring R.*

Proof. Let R be a right V-ring. Then by Corollary 7, $\underline{\mathfrak{CPI}}^{-1}(A) = \underline{\mathfrak{In}}^{-1}(A)$ for any R-module A. Hence A is indigent if and only if A is sc-indigent by [8, Proposition 5]. □

Proposition 34. *A module A is sc-indigent if and only if $\prod_{i\in I} A_i$ is sc-indigent where $A_i = A$ for all $i \in I$.*

Proof. Clear by Proposition 14. □

By Remark 28 and Proposition 34, sc-indigent rings are characterized as follows:

Corollary 35. *For a ring R, the following are equivalent:*

(1) *R_R is sc-indigent.*
(2) *Any direct product of copies of R is sc-indigent.*
(3) *Every free R-module is sc-indigent.*
(4) *There exists a cyclic projective sc-indigent R-module.*

Theorem 36. *Let R be a ring, B an R-module and A an R/I-module for any ideal I of R. If $B/BI \in \underline{\mathfrak{CPI}}^{-1}(A_{R/I})$, then $B \in \underline{\mathfrak{CPI}}^{-1}(A_R)$.*

Proof. Let $B/BI \in \underline{\mathfrak{CPI}}^{-1}(A_{R/I})$, and C be a copure extension of B and $g : B \to A$ an R-homomorphism. Since copure short exact sequences of R-modules form a proper class by [7, Proposition 8], B/BI can be embedded in C/CI as a copure submodule via $f : B/BI \to C/CI$ defined by $f(b + BI) = b + CI$ for any $b \in B$. Since $BI \subseteq Ker(g)$, there exists a homomorphism $h : B/BI \to A$ such that $h\pi_B = g$ where $\pi_B : B \to B/BI$. By assumption, there exists an R/I-homomorphism $\bar{h} : C/CI \to A$ such that $\bar{h}f = g$. Since h is also an R-homomorphism and $\bar{h}\pi_C i_B = g$ where $\pi_C : C \to C/CI$ and $i_B : B \to C$ is the inclusion. Thus $B \in \underline{\mathfrak{CPI}}^{-1}(A_R)$. ☐

Corollary 37. *Let I be an ideal of a ring R and A and B be R/I-modules. Then the following statements hold:*

(1) *$B \in \underline{\mathfrak{CPI}}^{-1}(A_R)$ if and only if $B \in \underline{\mathfrak{CPI}}^{-1}(A_{R/I})$.*
(2) *A is a copure-injective R-module if and only if A is a copure-injective R/I-module.*
(3) *A is an sc-indigent R-module if and only if A is an sc-indigent R/I-module.*

Proof. (1) If A_R is B-subcopure-injective, then clearly it is a B-subcopure-injective R/I-module. The converse follows by Theorem 36.
(2) By using Proposition 4, (2) follows from (1).
(3) Clear by (1) and (2). ☐

Recall [11] that a module A is called ps-poor if pure-subinjectivity domain of A consists of only pure-injective modules. Over a commutative classical ring R, by [8, Corollary 17], pure-injective modules and copure-injective modules coincide. Hence, the following result is immediate.

Proposition 38. *Let R be a commutative classical ring. Then an R-module A is sc-indigent if and only if A is ps-poor.*

Since by [16, Theorem 2] and [17, Proposition 4.1], every commutative (co-)noetherian ring is classical, we have the following result.

Corollary 39. *Let R be a commutative (co-)noetherian ring. Then an R-module A is sc-indigent if and only if A is ps-poor.*

Remark 40. *ps-poor abelian groups and sc-indigent abelian groups coincide by Corollary 39.*

Corollary 41. *Every finitely embedded $\mathbb{Z}$-module is copure-injective but not sc-indigent.*

Proof. Let A be a finitely embedded $\mathbb{Z}$-module. Then A is cofinitely related by [6, Proposition 17]. So A is copure-injective by [8, Proposition 3]. Since $\mathbb{Z}$ is not a CDS ring, by Proposition 29, A is not an sc-indigent module. ☐

Proposition 42. *If a ring R has an sc-indigent cc-injective module B, then every module with its copure injective extension has c.c cokernel is copure-injective.*

Proof. Let A be an R-module with the exact sequence $0 \to A \to C \to C/A \to 0$, where $A \to C$ is a copure extension of A with C is copure-injective. Consider the sequence $0 \to Hom(C/A, B) \to Hom(C, B) \to Hom(A, B) \to Ext^1(C/A, B)$. Since C/A is c.c., $Ext^1(C/A, B) = 0$. So by Lemma 3, $A \in \underline{\mathfrak{CPI}}^{-1}(B)$, that is A is copure-injective. □

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