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**On The Chebyshev Approximation by  $A + B^* \log(1 + CX)$**

by

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# **Communications de la Faculté des Sciences de l'Université d'Ankara**

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## On The Chebyshev Approximation by $A + B^* \log (1 + CX)$

A. ABDİK▲ and Ş. YÜKSEL▲▲

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### ABSTRACT

Previous studies on the Chebyshev approximation are enlightened, and the best Chebyshev approximation proved to be  $A + B^* \log (1 + CX)$  on  $[0, \alpha]$  and it is generalized with the help of new concepts.

### INTRODUCTION

The most general approximation problem, first presented in 1970 by Barrodal [1], can be express shortly as the following:

On the assumption that  $X$  is a topologic space and  $C(X)$  a set of bounded and continious functions (have real and complex values) on space  $X$ ,  $C(X)$  space can be set up by norm

$$\|g\| = \sup \{ |g(x)| ; x \in X \}$$

Let  $P$  be a parameter space and  $F$  approximation function in  $C(X)$  corresponding an element  $A$  of parameter space  $P$  such as  $F(A,.) = F[A]$ . There is an element,  $F[A]$ , for  $f$  which is in  $C(X)$  such that

$$\rho(f, X) = \inf \{ \|f - F[A^*]\| ; A \in P \}$$

with the condition of

$$\rho(f, X) = \|f - F[A^*]\|$$

then  $A$  is called "best parameter" and the function  $F[A^*]$  "best approximation" to  $f$  on  $X$ . Searching  $A^*$  is the essential of Chebyshev problem.

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Solution of Chebyshev approximation problem is carried out by means of varying  $X$ ,  $F$  and  $P$ . The conditions hold in for the solution of Chebyshev problem are important.

G. Meinardus and Schwedt [2] found out important theorems in 1964 which are used for the best approximation in Chebyshev problem. Then many scientists have studied on Chebyshev approximation problem under various conditions [3]. C.B. Dunham [4], [5] proved that the best approximation would be  $A+B*\log(1+CX)$  on  $[0, \alpha]$ .

In our study we set up new lemmas, theorems and definitions in order to enlighten the obscurities in previous studies and to prove the best Chebyshev approximation to be  $A+B*\log(1+CX)$  on  $[0, \alpha]$ . Furthermore, we have generalized it by means of new concepts.

#### EXTENSIVE SOLUTION OF CHEBYSHEV APPROXIMATION BY $A+B*\log(1+CX)$

Topologic concepts are invariant under an homomorphism.  $[-1, +1]$  is homomorph to  $[0, \alpha]$  so we can use  $[-1, +1]$  instead of  $[0, \alpha]$ .

Let  $C([-1, +1])$  be the space of defined and numerical functions on  $[-1, +1]$  with norm

$$\|g\| = \sup\{|g(x)|; -1 \leq x \leq +1\}$$

and with the condition

$$P = \{A: A = (a_1, a_2, a_3) \in \mathbb{R}^3\}$$

Consider the existence of approximation function  $F$ , corresponding to element  $f$  on the same space,  $C([-1, +1])$ . Let the approximation function has the form of

$$F(A, x) = a_1 + a_2 \log(1 + a_3 x)$$

for an element  $A$  of a selected parameter space,  $P$ . When  $\|a_3\| \geq 1$ ,  $\|F(A, \cdot)\|$  goes infinity so that the parameter  $a_3$  satisfies

$$-1 < a_3 < +1$$

After selecting an approximation function  $F$  as above, finding element  $A^*$  for which  $\|f - F(A, \cdot)\|$  is minimum, gives solution of

Chebyshev problem. Such an element  $A^*$  is called "best parameter" and  $F(A^*,.)$  "best approximation" to  $f$ .

We can put approximation functions of the type

$$F(A,x) = a_1 + a_2 \log(1 + a_3 x)$$

into two groups:

### 1. Constant approximation

Constant approximation is such approximation functions that correspond to parameters  $A = (a_1, 0, a_3)$  or  $A = (a_1, a_2, 0)$ . Really in this case  $F(A,x) = a_1$ .

### 2. Non-constant approximation

Now  $a_2 \neq 0$  and  $a_3 \neq 0$ , that is  $a_2 a_3 \neq 0$ . In this case approximation function is evidently unique.

**Lemma 1:** The difference between a constant approximation and another approximation has at most one zero in  $[-1, +1]$ .

**Proof:** Constant approximation is  $F(A,x) = a_1$  when  $A$  has the form  $A = (a_1, 0, a_3)$  or  $A = (a_1, a_2, 0)$ . Now, let non-constant another approximation function

$$F(B,x) = b_1 + b_2 \log(1 + b_3 x)$$

Due to the definition,  $b_2 b_3 \neq 0$ .

Consider that

$$d(x) = F(A,x) - F(B,x)$$

has two zeros in  $[-1, +1]$ . According to Rolle theorem

$$d'(x) = F'(A,x) - F'(B,x)$$

has zero at least for one  $x$  value. That is

$$d'(x) = - \frac{b_2 b_3}{1 + b_3 x} = 0$$

This implies  $b_2 = 0$  or  $b_3 = 0$ . However, this is a contradiction to the assumption that  $b_2 b_3 \neq 0$ .

**Lemma 2:** The difference between a non-constant approximation and a linear approximation has at most two zeros in  $[-1, +1]$ .

**Proof:** Under the circumstances of  $-1 < a_3 < +1$ , consider the difference between

$$F(A, x) = a_1 + a_2 \log(1 + a_3 x) \text{ and } a_4 + a_5 x$$

Suppose  $d(x) = F(a, x) - a_4 - a_5 x$  has three zeros in  $[-1, +1]$ .

Then derivative of  $d(x)$ ,

$$d'(x) = \frac{a_2 a_3 - a_5 - a_5 a_3 x}{1 + a_3 x}$$

has at most zeros in  $[-1, +1]$ .

For the approximation function,  $F(A, x) = a_1 + a_2 \log(1 + a_3 x)$ , to be definite in  $[-1, +1]$ ,  $1 + a_3 x > 0$  is required. Then the right hand side of

$$(1 + a_3 x) d'(x) = a_2 a_3 - a_5 - a_5 a_3 x$$

is a polynomial of first degree and has at most one zero. On the other hand if  $d'$  is identically zero then

$$a_2 a_3 - a_5 = 0$$

and

$$a_2 a_3 = 0$$

$F(A,.)$  is another non-constant approximation, so  $a_2 a_3 \neq 0$ . Then  $a_5 = 0$ . Inserting this value in the above equation we have  $a_2 a_3 = 0$ . However, this a contradiction to the non-constant approximation,  $F(A,.)$ .

**Lemma 3:** The difference between a non-constant approximation and another approximation has at most two zeros in  $[-1, +1]$ .

**Proof:** Let  $F(A,.)$  and  $F(B,.)$  be two non-constant approximation functions.

Suppose  $d(x) = F(A, x) - F(B, x)$  has three zeros, so  $d'(x)$  has the form of

$$d'(x) = F'(A, x) - F'(B, x) = \frac{(a_2 a_3 - b_2 b_3) + (a_2 a_3 b_3 - a_3 b_2 b_3)x}{(1 + a_3 x)(1 + b_3 x)}$$

which must have at most two zeros.  $F(A, x)$  and  $F(B, x)$  to be definite in  $[-1, +1]$  so that  $1 + a_3 x > 0$  and  $1 + b_3 x > 0$  are required. Then the right hand side of

$(1+a_3x)(1+b_3x)d'(x) = (a_2a_3-b_2b_3) + (a_2a_3b_3-a_3b_2b_3)x$   
 is a polynomial of the first degree so that it has at most one zero  
 and then  $d$  has at most two zeros.

On the other hand if  $d'$  is identical to zero,  $d$  must be constant. In that case  $d$  has zeros if and only if  $d'=0$ . This is a contradiction. More clearly

$$a_2a_3 - b_2b_3 = 0$$

and

$$a_3b_3(a_2 - b_2) = 0$$

are required. Approximation functions are not constant, hence  $a_2a_3 \neq 0$  and  $b_2b_3 \neq 0$ . From the second equation we find  $a_2 = b_2$  and inserting it in the first equation we have  $a_3 = b_3$  and  $d = a_1 - b_1$ . Here again if  $d$  has zeros which imply  $a_1 = b_1$  then we get  $F(A,.) = F(B,.)$  which contradicts the assumption.

**Definition 1:** Define linear space  $D(A,...)$  formed by  $\partial F(A,.) / \partial a_i$ , where  $i=1, 2, 3$  and let the dimension be  $d(A)$ . Then  $d(A)$  evidently depends on  $A$ .

If each non-zero element of linear space  $D(A,...)$  has at most  $d(A)-1$  zeros at element  $B$  of parameter space  $P$  then the space  $D(A,...)$  has "Classical HAAR" property.

A linear space that has the property of classical Haar is called Haar subspace.

**Lemma 4:** If  $D(A,...)$  correspond a constant approximation then exists a parameter  $A$  with a Haar subspace of dimension two.

**Proof:** Let  $A=(a_1, a_2, a_3)$ , then it has continuous derivatives,  $\partial F(A,x) / \partial a_i$ :

$$\frac{\partial F(A,x)}{\partial a_1} = 1 ; \quad \frac{\partial F(A,x)}{\partial a_2} = \log(1+a_3x) ; \quad \frac{\partial F(A,x)}{\partial a_3} = \frac{a_2x}{1+a_3x}$$

Let  $B=(b_1, b_2, b_3)$ , then an element of  $D(A,...)$  has the following form,

$$D(A,B,x) = \sum_{i=1}^3 b_i \frac{\partial F(A,x)}{\partial a_i} = b_1 + b_2 \log(1+a_3x) + b_3 \frac{a_2x}{1+a_3x}$$

If we select the approximation function  $F(A,.)$  as constant and take  $A=(a_1,0, a_3)$  then we have

$$D(A,B,x) = b_1 + b_2 \log(1 + a_3 x)$$

It is evidently seen that  $D(A,B,x)$  is an element of linear space of two dimentionions.

On the other hand,  $D(A,B,x)$  has at most one zero in  $[-1, +1]$  according to Lemma 1, under the condition that  $D(A,B,x) \neq 0$ .

In that case,  $D(A,.,.)$  is an "Haar subspace" of two dimentionions for  $A = (a_1,0, a_3)$ .

**Lemma 5:** If  $F(A,.)$  is any non-constant approximation then  $D(A,.,.)$  is a Haar subspace of dimention 3.

**Proof:** Since the approximation function  $F(A,.)$  is non-constant  $a_2$  and  $a_3$  are non-zero and

$$D(A,B,x) = b_1 + b_2 \log(1 + a_3 x) + b_3 \frac{a_2 x}{1 + a_3 x}$$

is clearly an element of vector space of dimention 3. This shows that  $D(A,.,.)$  is a linear vector space of dimention 3.

Let  $D(A,B,x)$  be a non-zero element of  $D(A,.,.)$  then  $B = (b_1, b_2, b_3) \neq 0$ . Since

$$D'(A,B,x) = \frac{(b_2 a_3 + b_3 a_2) + b_2 a_3^2 x}{(1 + a_3 x)^2}$$

has at most one zero in  $[-1, +1]$  then  $D(A,B,x)$  has at most two zeros. On the other hand since  $D'(A,B,x) = 0$  then  $b_2 a_3 + b_3 a_2 = 0$  and  $b_2 a_3^2 = 0$ . Using  $a_2 \neq 0$  and  $a_3 \neq 0$  circumstances, we have  $b_2 = 0$  and  $b_3 = 0$ . That is

$$D(A,B,x) = b_1$$

From the assumption  $B = (b_1, b_2, b_3) \neq 0$  it is necessary to be  $b_1 \neq 0$ . In that case  $D(A,.,.)$  is a Haar subspace of dimention 3.

**Remark 1:** If  $A$  corresponds to a constant approximation function, Lemma 1 shows that  $d(A)=2$ . Otherwise Lemma 3 gives  $d(A)=3$ .



Now, to obtain a result of DE LA VALLEE-POUSSIN type which is useful in characterizing "near best approximation", let us consider a compact-Hausdorff space,  $X$  and prove some theorems.

Let us consider a compact Hausdorff space  $X$ , and a set  $C(X)$  of all continuous functions on  $X$ . If  $P$  be a parameter space and  $f$  be any element of  $C(X)$  then  $S(A,B;x)$  is defined such as

$$S(A,B,x) = (F(A,x) - f(x)) (F(A,x) - F(B,x))$$

where  $A$  and  $B$  are elements of  $P$ . Now, let us prove that

$$\rho(f) = \inf \{ \| F(A,.) - f \| ; A \in P \}$$

has a sublimit.

**Theorem 1:** Let  $A$  be an element of parameter space,  $P$ . If for each element,  $B$ , of  $P$ , there is a closed subset,  $K$ , of  $X$  such that

$$\min \{ S(A,B;x) ; x \in K \} \leq 0$$

then

$$\rho(f) \geq \min \{ |F(A,x) - f(x)| ; x \in K \} = \sigma$$

**Proof:** Suppose  $\rho(f) < \sigma$  then

$$\rho(f) < \| F(B,.) - f \| < \sigma$$

such that there exists an element,  $B$ , of  $P$ . Hence for the elements  $x$  of  $K$

$$| F(A,x) - f(x) | - | F(B,x) - f(x) | > 0$$

and

$$\begin{aligned} S(A,B,x) &= | F(A,x) - f(x) |^2 - (F(A,x) - f(x))(F(B,x) - f(x)) \\ &\geq | F(A,x) - f(x) | (| F(A,x) - f(x) | - | F(B,x) - f(x) |) > 0 \end{aligned}$$

This contradicts the hypothesis.

**Definition 2:** For a  $g$  element of space  $C([-1, +1])$  if there exist

$$|g(x_i)| = \|g\|, \quad g(x_i) = (-1)^i g(x_i); \quad (i = 1, 2, \dots, d(A))$$

and point set  $\{x_1, x_2, \dots, x_{d(A)+1}\}$  such that  $-1 \leq x_1 < \dots < x_{d(A)+1} \leq +1$  then  $g$  function alternates  $d(A)$  times.

**Theorem 2:** If approximation function  $F$  has property (Z) at  $A$  and for an element  $f$  of  $C([-1, +1])$ ,  $F(A,.) - f$  alternates on  $\{x_1, x_2, \dots, x_{d(A)+1}\}$  then there exists property

$$\rho(f) \geq \min\{|FA, x_k) - f(x_k)| : 1 \leq k \leq d(A) + 1\}$$

**Proof:** Since the function  $F(A, \cdot) - f$  changes alternatively on  $\{x_1, x_2, \dots, x_{d(A)+1}\}$ , there exists the property

$$\text{Sgn}(F(A, x_j) - f(x_j)) = -\text{Sgn}(F(A, x_{j+1}) - f(x_{j+1})) \quad (1)$$

where,  $j = 1, 2, \dots, d(A)$

Let  $K$  in theorem 1 as  $K = \{x_k; 1 \leq k \leq d(A) + 1\}$  then one gets

$$\rho(f) \geq \min\{|F(A, x_k) - f(x_k)| : 1 \leq k \leq d(A) + 1\}$$

In that case at least for an  $x_p \in K$ , one gets

$$S(A, B, x_p) = (F(A, x_p) - f(x_p)) (F(A, x_p) - F(B, x_p)) \leq 0$$

Otherwise  $F(A, \cdot) - F(B, \cdot)$  has  $d(A) + 1$  zeros in  $[-1, +1]$  according to the property (1). This contradicts the hypothesis that  $F(A, \cdot)$  has property (Z) at  $A$ .

**Definition 3:** Approximation function  $F(A, \cdot)$  has the property of local Haar space, with null points of degree  $d(A)$  at  $A$ , if the following conditions are fulfilled:

(I) Approximation function  $F(A, \cdot)$  has continuous partial derivatives for each  $i$ ,  $i = 1, 2, \dots, n$ .

(II) Setting

$$D(A, B, x) = (B, \nabla F(A, x)) = \sum_{i=1}^n b_i \frac{\partial F(A, x)}{\partial a_i}$$

we have

$$F(A + B, x) - F(A, x) = D(A, B, x) + R(A, B, x)$$

and when  $\|B\|$  is sufficiently small

$$R(A, B, x) = O(\|B\|)$$

(III) There exists a neighbourhood of element  $A$  which is contained in  $P$ .

(IV) Linear space  $D(A, \cdot, \cdot)$  is a Haar subspace of dimension  $d(A)$  in  $[-1, +1]$ .

**Remark 2:** Approximation function  $F$  has local Haar space condition, only when  $D(A, \cdot, \cdot)$  obeys classical Haar condition.

**Theorem 3:** If approximation function  $F$  has the local property with null points of degree  $d(A)$  at  $A$  and function  $f$  be

an element of space  $C([-1, +1])$ , and  $F(A, \cdot)$  be the best approximation to  $f$ , then function  $F(A, \cdot) - f$  alternates  $d(A)$  times.

**Proof:** Let  $F$  be the best approximation to  $f$ , then set of extreme points of  $F(A, x) - f(x)$ ,

$M_A = \{x / x \in [-1, +1] : \|F(A, \cdot) - f\| = |F(A, x) - f(x)|\}$  has at least  $d(A) + 1$  elements.

Under the above conditions there exist some points which hold  $-1 \leq x_1 < \dots < x_{d(A)+1} < +1$  and set  $\{x_1, x_2, \dots, x_{d(A)+1}\}$  is an alternant of  $F(A, \cdot) - f$ . Otherwise there would be found a natural number,  $m$ , and so we can separate  $[-1, +1]$  into  $m + 1$  subintervals such that each interval contains an extreme point and  $F(A, x) - f(x)$  has same sign in these intervals.

The set of extreme points of  $F(A, x) - f(x)$ , has  $d(A) + 1$  elements, hence, for  $k = 1, 2, \dots, d(A)$ , a non-zero element  $B$  of parameter space  $d'$  can be found [2] such that

$$(B, \nabla F(A, x_k)) = \sum_{i=1}^n b_i \frac{\partial F(A, x_k)}{\partial a_i} - F(A, x_k) - f(x_k)$$

and so for all extreme points,  $x$ ,

$$(F(A, x) - f(x)) (B, \nabla F(A, x)) = |F(A, x) - f(x)|^2$$

and then

$$\text{Sgn}(B, \nabla F(A, x)) = \text{Sgn}(F(A, x) - f(x))$$

This result contradicts the hypothesis of the best approximation function  $F$  to  $f$ .

Meinardus and Schwedt ([2] theorem 9) showed that a set  $M_A$  of extreme points, has at most  $d(A) + 1$  points in  $[-1, +1]$ .

Opposition of the Theorem 2 is correct, provided the above conditions are taken into account.

Now, combining Theorem 2 and Theorem 3 one can get the following result:

**Theorem 4:** If  $F(A, \cdot)$  has property (Z) at  $A$  and local Haar property with null points of degree  $d(A)$ , then  $F(A, \cdot)$  be the best to  $f$  if and only if  $F(A, \cdot) - f$  alternates  $d(A)$  times.

**Theorem 5:** If  $F(A,.)$  satisfies the condition of Theorem 4, and  $F(A,.)$  is best, then it is a unique best approximation.

**Proof:** Suppose,  $F(A,.)$  and  $F(B,.)$  are two approximation functions. We can take  $d(A) \leq d(B)$ , without violating the generality.

Let set of extreme points of  $F(A,.) - f$  be  $\{x_1, x_2, \dots, x_{d(A)+1}\}$  ( $k = 1, 2, \dots, d(A)+1$ ). According to Theorem 3, the set  $\{x_1, x_2, \dots, x_{d(A)+1}\}$  is an alternant of  $F(A,.) - f$ . Then we have

$$F(A, x_{j+1}) - f(x_{j+1}) = - (F(A, x_j) - f(x_j))$$

where,  $j = 1, 2, \dots, d(A)$ . Hence using Equation 1 we get inequalities system

$$F(A, x_1) - F(B, x_1) \leq 0$$

$$F(A, x_2) - F(B, x_2) \geq 0$$

or

$$F(A, x_1) - F(B, x_1) \geq 0$$

$$F(A, x_2) - F(B, x_2) \leq 0$$

.....

It is sufficient to investigate the first part,

$$F(A, x_1) - F(B, x_1) \leq 0$$

$$F(A, x_2) - F(B, x_2) \geq 0$$

.....

If the inequalities had been certain,  $F(A,.) - F(B,.)$  would have had  $d(A)+1$  definite null points and from the Haar condition we would have gotten result

$$F(.,) = F(B,.)$$

On the other hand, if the inequalities had been correct for a  $k_0$ , we would have gotten

$$F(A, x_{k_0}) - F(B, x_{k_0}) \neq 0$$

$$\text{Sng } (F(A, x_{k_0}) - F(B, x_{k_0})) = (-1)^{k_0}$$

However, if  $F(.,)$  and  $F(B,.)$  are two approximation functions and if we take

$$A(t) = (1-t) A + t B$$

$$B(t) = (1-t) B + t A$$

then  $F(A(t),.)$  and  $F(B(t),.)$  are also approximation functions. If we denote  $\delta = B - A$  in

$$B(t) = B - t (B - A)$$

we get

$$B(t) = B - t \delta$$

where, parameter  $\delta$  is an element of space  $p$ .

Since  $D(B,.,.)$  satisfies Haar condition, each non-zero element of  $D(B,.,.)$  has at most  $d(A)-1$  null points at element  $\delta$  of parameter space  $P$ . So  $F(B,.)$  have local Haar property.

Using property (II) of local Haar condition in  $F(B,x) - F(B-t\delta,x)$  we get

$$F(B,x) - F(B-t\delta,x) = tD(B,\delta,x) + R(B,\delta,x)$$

and adding the approximation function  $F(A,.)$  to the each side of this equation and denoting  $R(B,\delta, x) = 0(t)$ , we find

$$F(A,x) - F(B-t\delta,x) = F(A,x) - F(B,x) + tD(B,\delta,x) + 0(t)$$

We get the following system, for  $t > 0$ ,

$$F(A,x_1) - F(B-t\delta,x_2) < 0$$

$$F(A,x_2) - F(B-t\delta,x_2) > 0$$

.....

Thus  $F(A,.) - F(B-t\delta,.)$  has at least  $d(A)$  null points in  $[-1, +1]$  and when  $t$  is approaching to zero we get

$$F(A,.) = F(.,.)$$

### ÖZET

Chebyshev yaklaşımı üzerine daha önce yapılan çalışmalar aydınlatılmış,  $[0,\alpha]$  üzerine en iyi Chebyshev yaklaşımının  $A+B^* \log(1+CX)$  olduğu ispatlanmış ve yeni kavramlar yardımıyla konu geliştirilmiştir.

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