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Comments and Addenda

Internal structure of composite systems described by infinite-component wave equations*

I. H. Duru
Faculty of Sciences, University of Ankara, Turkey

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It is shown that the boson representation of infinite-component SO(4,2) wave equations represents a system of four oscillators as well as three-dimensional Kepler dynamics. These two pictures are complementary to each other; that is, the Kepler dynamics is adequate for small distances, while the oscillator dynamics is more relevant for infinitely large distances, and both pictures are equivalently possible for finite distances. This is interpreted as a sort of constituent confinement.

I. INTRODUCTION

Infinite-component wave equations describe composite systems in a relativistically invariant manner. The Majorana equation[1] based on the group SO(3, 2) and the equations based on the group SO(4, 2),[2] are well-known examples of these kinds of equations. Some years ago it was shown that one can introduce internal coordinates into the Majorana equation such that the Majorana system looks either like a two-dimensional oscillator, or like a two-dimensional Kepler problem.[3] Because of its group structure the Majorana system has only two degrees of freedom. For the generalized infinite-component wave equations, with the larger group SO(4, 2), it is possible to describe three-dimensional dynamical systems such as the H atom in three dimensions. In fact, the SO(4, 2) representation which implies internal Kepler coordinates has been in use for some time.[4] In this paper it is shown that internal oscillator coordinates can also be introduced to an infinite-component SO(4, 2) wave equation and the transformation from four-dimensional space to three-dimensional space is given for the boson representation. Then we see that for the boson representation an SO(4, 2) wave equation describes either the three-dimensional Kepler problem, or equally a system of four oscillators. These four oscillators (or constituents) oscillate on the surface of a sphere whose radius is proportional to $r^{1/2}$ [see Eq. (21)], where r is the relative distance between the constituents in the Kepler picture. However, the transformation from the four-oscillator (or constitutents) system to the three-dimensional Kepler system is possible only for finite size of the composite system. When the distance between the constituents goes to infinity (i.e., $r\to\infty$) only the oscillator system is possible; in other words, for $r\to\infty$ the internal potential is proportional to $\sum_{i=1}^{4} x_i^2 \simeq r$, not to $1/r$. Thus the oscillators (or constituents) are confined to a sphere of radius $s_0(r/2r_0)^{1/2}$ [Eq. (21)]. The spherical surface oscillates in the radial direction. On the other hand, for $r/r_0 \ll 1$ (but $r>0$) the Kepler picture (i.e., $1/r$ potential) is dominant, because the oscillator potential $\sum_{i=1}^{4} x_i^2 \simeq r$ becomes very small.

Note that only one of these two internal dynamical pictures permits the minimal coupling of the constituents with the external electromagnetic field. Minimal coupling is possible for the internal Kepler coordinates but not for the oscillator (or constituents) system.[5] This is in agreement with the quark-confinement consideration, because if a constituent particle is bound strongly to the system by an oscillator potential it cannot behave as a bare particle, and thus does not couple minimally to the external field.

II. THE INFINITE-COMPONENT SO(4,2) WAVE EQUATION

The representation of the Lie algebra of the group SO(4, 2) in terms of creation and annihilation operators is given by (Ref. 2)

$$\begin{aligned}
L_{ij} &= \tfrac{1}{2}\epsilon_{ijk}(a^\dagger\sigma_k a + b^\dagger\sigma_k b) \equiv L_k\,,\\
L_{i4} &= -\tfrac{1}{2}(a^\dagger\sigma_i a - b^\dagger\sigma_i b) \equiv A_i\,,\\
L_{i5} &= -\tfrac{1}{2}(a^\dagger\sigma_i C b^\dagger - aC\sigma_i b) \equiv M_i\,,\\
L_{i6} &= \frac{1}{2i}(a^\dagger\sigma_i C b^\dagger + aC\sigma_i b) \equiv \Gamma_i\,,\\
L_{46} &= \tfrac{1}{2}(a^\dagger C b^\dagger + aCb) \equiv S \equiv \Gamma_4\,,\\
L_{45} &= \frac{1}{2i}(a^\dagger C b^\dagger - aCb) \equiv T\,,\\
L_{56} &= \tfrac{1}{2}(a^\dagger a + b^\dagger b + 2) \equiv \Gamma_0\,,
\end{aligned} \tag{1}$$

where

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i = 1, 2, 3 \tag{2}$$

and a_i and b_i are the boson operators satisfying

$$\begin{aligned}
[a_i, a_j^*] &= [b_i, b_j^*] = \delta_{ij}\,,\\
[a_i, b_j] &= [a_i, b_j^*] = 0\,,
\end{aligned} \tag{3}$$

and

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

The generators $(L_{ab} = -L_{ba})$, $a, b = 1, 2, 3, 4, 5, 6$ with the metric $(-----++)$, satisfy the commutation relations

$$[L_{ab}, L_{cd}] = -i(g_{ac}L_{bd} + g_{bd}L_{ac} - g_{ad}L_{bc} - g_{bc}L_{ad})\,. \tag{4}$$

In the representation given by Eq. (1) L_k are the angular momentum operators, A_i are the analogs of the Lenz vector, M_i are the Lorentz boosts, $\Gamma_\mu = (\Gamma_0, \Gamma_i)$ is a four-vector, S is a Lorentz scalar operator, and T is the rotational-scalar-tilt operator.

We can also realize the representation given by Eq. (1) in terms of the complex numbers

$$\begin{aligned}
z &= \frac{1}{s_0}(x_1 + ix_2)\,, \quad \bar{z} = \frac{1}{s_0}(x_1 - ix_2)\,, \quad \partial_z = \frac{\partial}{\partial z}\,, \quad \bar{\partial}_z = \frac{\partial}{\partial \bar{z}}\,,\\
\eta &= \frac{1}{s_0}(x_3 + ix_4)\,, \quad \bar{\eta} = \frac{1}{s_0}(x_3 - ix_4)\,, \quad \partial_\eta = \frac{\partial}{\partial \eta}\,, \quad \bar{\partial}_\eta = \frac{\partial}{\partial \bar{\eta}}
\end{aligned} \tag{5}$$

which satisfy the commutation relations

$$\begin{aligned}
[\partial_z, z] &= [\bar{\partial}_z, \bar{z}] = 1\,,\\
[\partial_\eta, \eta] &= [\bar{\partial}_\eta, \bar{\eta}] = 1
\end{aligned} \tag{6}$$

and all the other operators of Eq. (5) commute with each other. s_0 in Eq. (5) carries a unit of length so that z and η are dimensionless. The relation of the complex numbers with the boson operators is given by

$$\begin{aligned}
a_1 &= \frac{1}{\sqrt{2}}(\bar{z} + \partial_z)\,, \quad a_2 = \frac{1}{\sqrt{2}}(z + \bar{\partial}_z)\,,\\
a_1^* &= \frac{1}{\sqrt{2}}(z - \partial_z)\,, \quad a_2^* = \frac{1}{\sqrt{2}}(\bar{z} - \partial_z)
\end{aligned} \tag{7a}$$

and

$$\begin{aligned}
b_1 &= \frac{1}{\sqrt{2}}(\bar{\eta} + \partial_\eta)\,, \quad b_2 = \frac{1}{\sqrt{2}}(\eta + \bar{\partial}_\eta)\,,\\
b_1^* &= \frac{1}{\sqrt{2}}(\eta - \bar{\partial}_\eta)\,, \quad b_2^* = \frac{1}{\sqrt{2}}(\bar{\eta} - \partial_\eta)\,.
\end{aligned} \tag{7b}$$

Then the generator Γ_0 of SO(4, 2) which we will diagonalize takes the form

$$\Gamma_0 = \tfrac{1}{2}[(\eta\bar{\eta} + z\bar{z}) - (\partial_\eta\bar{\partial}_\eta + \partial_z\bar{\partial}_z)]\,. \tag{8}$$

The scalar product for the complex-number representation is given by

$$(f, g) = \frac{1}{\pi}\int \bar{f}(z, \bar{z}; \eta, \bar{\eta})\, g(z, \bar{z}; \eta, \bar{\eta})(\tfrac{1}{2}i)^2 dz\, d\bar{z}\, d\eta\, d\bar{\eta}\,. \tag{9}$$

For these representations the Casimir operators have the values

$$\begin{aligned}
Q_2 &= L_{ab}L^{ab} = -3\,,\\
Q_3 &= \epsilon_{abcdef}L^{ab}L^{cd}L^{ef} = 0\,,\\
Q_4 &= L_{ab}L^{bc}L_{cd}L^{da} = 0\,.
\end{aligned} \tag{10}$$

If we diagonalize $L_{56} = \Gamma_0$ and the $O(4) \supset O(3)$ subgroup chain, we obtain the angular momentum states $|nlm\rangle$, where n, l, and m are the eigenvalues of Γ_0, L, and L_z, respectively. When the generators given by Eq. (1) act on the vacuum states, we get the boson tower with $l = 0, 1, 2, \ldots$ and, when they act on the states $a^\dagger|0\rangle$ and $b^\dagger|0\rangle$, we get the fermion tower with $l = \frac{1}{2}, \frac{3}{2}, \ldots$.[6]

The simplest infinite-component SO(4, 2) wave equation with the current $J_\mu = \Gamma_\mu$ is

$$(P^\mu\Gamma_\mu - K)\psi(P) = 0\,. \tag{11}$$

In the rest frame of the particle we obtain

$$(M\Gamma_0 - K)\psi(0) = 0\,, \tag{12}$$

and since the spectrum of Γ_0 is n, where $n = 1, 2, 3, \ldots$ for the boson representation, and $n = \frac{3}{2}, \frac{5}{2}, \ldots$ for the fermion representation, this equation gives

$$M = K/n \tag{13}$$

for the mass spectrum, which is of the Majorana type.[1] The general form of the SO(4, 2) wave equation gives a rising spectrum,[7] applicable to composite systems, but for the purpose of demonstrating the two types of internal structures the equation given by (11) is adequate.

III. REPRESENTATION BY DIAGONAL OSCILLATOR COORDINATES

In the space of z and η given by Eq. (5),

$$z\bar{z}=\frac{1}{s_0^2}(x_1^2+x_2^2)\,,\quad \eta\bar{\eta}=\frac{1}{s_0^2}(x_3^2+x_4^2)\,,$$
$$\partial_z\bar{\partial}_z=\frac{s_0^2}{4}(\partial_{x_1}^2+\partial_{x_2}^2)\,,\quad \partial_\eta\bar{\partial}_\eta=\frac{s_0^2}{4}(\partial_{x_3}^2+\partial_{x_4}^2)\,. \tag{14}$$

Then the operator Γ_0 of (8) becomes

$$\Gamma_0=\frac{1}{2}\left(\frac{1}{s_0^2}\sum_{i=1}^{4} x_i^2-\frac{s_0^2}{4}\sum_{i=1}^{4}\partial_{x_i}^2\right), \tag{15}$$

and the rest-frame equation (12) takes the form

$$\left(-\frac{1}{2}\sum_{i=1}^{4}\partial_{x_i}^2+\frac{1}{2}\frac{4}{s_0^4}\sum_{i=1}^{4}x_i^2-\frac{4K}{Ms_0^2}\right)\psi(0)=0\,.$$

In order to make the units correct we multiply this equation by $\hbar^2/\mu$:

$$\left(-\frac{\hbar^2}{2\mu}\sum_{i=1}^{4}\partial_{x_i}^2+\tfrac{1}{2}\mu\omega^2\sum_{i=1}^{4}x_i^2-E\right)\psi(0)=0\,, \tag{16}$$

where

$$\omega=\frac{2\hbar}{\mu s_0^2}\,,\quad E=\frac{4\hbar^2K}{\mu M s_0^2}\,. \tag{17}$$

Equation (16) is the equation of four uncoupled oscillators oscillating with the same frequency in the center-of-mass frame of the composite particle, and it has the spectrum

$$E_{n'}=\frac{2\hbar^2}{\mu s_0^2}(n'+2)\,,\quad n'=0,1,2,\ \ldots\,. \tag{18}$$

By comparing this with Eq. (17) we obtain

$$M=K/(\tfrac{1}{2}n'+1)$$

or

$$M=\frac{K}{n}\ \text{ for }\ n=1,\tfrac{3}{2},2,\tfrac{5}{2},\ \ldots \tag{19}$$

which is equal to the spectrum given by Eq. (13).

IV. REPRESENTATION BY DIAGONAL KEPLER COORDINATES

Now we pass to an r space by the transformation

$$z=\left(\frac{r}{2r_0}\right)^{1/2}\sin\tfrac{1}{2}\theta e^{i\varphi}\,,$$
$$\eta=\left(\frac{r}{2r_0}\right)^{1/2}\cos\tfrac{1}{2}\theta e^{i\varphi}\,; \tag{20}$$

where r_0 carries the dimension of length. This transformation implies that

$$z\bar{z}+\eta\bar{\eta}=\frac{r}{2r_0}$$

or

$$\frac{1}{s_0^2}\sum_{i=1}^{4}x_i^2=\frac{r}{2r_0}\,. \tag{21}$$

That is, the oscillators of the previous representation are constrained to oscillate on the surface of a sphere of radius $s_0(r/2r_0)^{1/2}$. Thus the oscillator system is like a sphere oscillating in the radial direction. The transformation of (20) is a dynamical restriction, which excludes the fermion representation of SO(4,2), as will be seen in Eq. (25). In terms of these new coordinates in r space, the operator Γ_0 becomes

$$\Gamma_0=-rr_0\nabla^2+\frac{r}{4r_0}\,, \tag{22}$$

where

$$rr_0\nabla^2=\tfrac{1}{2}(\partial_z\bar{\partial}_z+\partial_\eta\bar{\partial}_\eta)$$
$$=rr_0\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right].$$

Thus, after multiplication from the left by $1/r$ (for $r\neq 0$ and to be finite) and by $\hbar^2/2\mu$, the rest-frame equation of (12) becomes

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2-\frac{\alpha}{r}-E\right)\psi(0)=0\,, \tag{23}$$

where

$$\alpha=\frac{\hbar^2K}{2\mu M r_0}\,,\quad E=-\frac{\hbar^2}{8\mu r_0^2}\,. \tag{24}$$

Equation (23) represents a three-dimensional Kepler motion with energy spectrum

$$E=-\frac{\mu\alpha^2}{2\hbar^2}\frac{1}{n^2}\,,\quad n=1,2,\ \ldots\,. \tag{25}$$

From Eq. (24) we get back the original mass spectrum of bosons, i.e., $M=K/n$, $n=1,2,\ \ldots\,,$ not of fermions. The exclusion of fermions, as we mentioned previously, is due to the transformation (21). We cannot describe a three-dimensional system with spin in terms of three space coordinates only; for that we need a fourth degree of freedom, independent of space coordinates.

From Eq. (21) we see that for $r/r_0 \ll 1$, the total oscillator potential $V_{\text{osc}} \simeq \sum_{i=1}^{4} x_i^2$ becomes very small. Thus for very close distances (yet still $r > 0$) the adequate picture is the internal Kepler motion. On the other hand, as $r \rightarrow \infty$ we cannot obtain Eq. (23), and we have only the internal oscillator motion.

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