Establishing Conceptual Bases for the Measurement of Volume

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Abstract. Fourth grade students' understanding of rectangular solids made of small cubes was investigated. A three-phase procedure was utilized. First, interviews were conducted individually to assess the students' level of functioning in cube-enumeration tasks. Second, participants were engaged in equal sharing of spatial constructions. Last, postinterviews were conducted to probe students' improvements as revealed by their use of enumeration strategies. Students used three distinct conceptualizations for the arrays of cubes depending on what they formed as unit and how they structured the whole building. Initially, their structuring was distracted by the complexity of buildings and none of them used the same strategies consistently across problems. During the instruction, they exhibited the same conceptualizations and transitioned from one to the other. After the intervention, all the students consistently used layering strategies regardless of the complexity of the buildings. Equal-sharing situations coupled with coloring activities paved the road in establishing units, composite units, and unit iteration.

Introduction

Meaningful mathematical learning occurs when the learner understands the reason behind the procedures. For example, children should come to understand the logical operation (i.e., organization of units) of finding the volume before they are introduced to the numerical calculation or formula (Piaget, Inhelder, & Szeminska, 1970). Gradually, they substitute numerical operations for logical operations. In this sense, finding the number of cubes in rectangular solids provides the cognitive framework for understanding the measurement of volume and the formula for determining the volume (Battista & Clements, 1998). However, children have difficulty finding the number of cubes in rectangular solids even in high school and beyond (Battista & Clements, 1996; Ben-Chaim, Lappan, & Houang, 1985). In order to overcome these difficulties, children need ample experiences with appropriate instructional materials that engage them in mathematical discovery.

For any mathematical concept, students pass through various levels of understanding. As mathematics education researchers, we need to clarify what these levels are and how they are attained by the students as well as what we can do to help them gain a deeper understanding of the concept. Previously, researchers have described the cognitive constructions students make as they enumerate 3-D arrays of cubes (Battista & Clements, 1996). The present study aims at extending previous research about students' understanding of rectangular solids made of small cubes and attempts to extend the research base by utilizing different problem situations, such as equal sharing of spatial constructions, to explore students' thinking.

There were two main reasons why the equal sharing context was used. First, it is a semantically rich context (Empson, 1995) that could initiate students' intuition and prior knowledge about sharing things. While solving problems using their intuition and prior knowledge, children discover new ways of looking at things, thus leading to the development of that knowledge into more sophisticated ones. For example, layer structure is not inherent in 3-D arrays of cubes (Cobb, Yackel, & Wood, 1992). Children gradually and individually construct the layer structure while they are solving problems and acting on similar objects involving layers. Second, equal sharing provides a socially desirable context in which students want to be fair in their partitioning of the buildings. This might lead them to abstract the equality of the spatial structural elements of the buildings such as layers, columns, and rows as iterable units.

Another component of the present study was the utilization of social interaction between peers and the teacher. Through interaction with peers and adults students appropriate their knowledge and come to a common understanding of the problem situation, which is necessary for communicating mathematically. Additionally, asking questions that constrain students to direct their attention to certain aspects of the problem might be a good strategy for stimulating learning (Anderson, Reder, & Simon, 1995).

Methods

Participants

Four fourth graders, two girls and two boys from a charter school in a poor socioeconomic area in the southwest of America , participated in the study. According to their classroom teacher, they were all average or slightly below average students in their mathematics classes. The students participated in the study on a voluntary basis. They said they had not previously seen the same materials used in the study.

Materials

Four types of materials were used: colored pens, wooden cubes (two centimeters along each edge), rectangular buildings made by gluing individual wooden cubes together (the same sizes as the loose cubes), and drawings of the concrete buildings.

Pre- and postinterview tasks consisted of four buildings composed of 4, 8, 16, and 36 cubes, and their parallel-perspective drawings. In all of the pre and postinterview tasks, students were asked to find the number of small (unit) cubes in the buildings. Colored pens were used by the participants to show the different shares by shading parts of the buildings in drawings during the intervention period. The loose wooden cubes were used for constructing the buildings. Seven rectangular buildings composed of a varying number of cubes, that is, 8, 9, 12, 16, 24, 36, and 48 cubes were used as a supplement to the pictorial representations. Drawings of these buildings were used by the participants for shading in different shares and comparing with the concrete buildings.

Procedures

A three-phase procedure was applied for collecting the data. In the first phase, the interviews were intended to determine the levels of students' thinking in cube enumeration tasks. Participants were asked to find the number of cubes in rectangular buildings using first, graphical and then, concrete representations of rectangular buildings made of small cubes. They were asked to solve the problems with concrete cubes after they finished all the problems with pictorial representations. In volume-related tasks, although the differences between the modes were not statistically significant, the third and fifth graders' overall scores in problems with concrete representations were slightly better than with the graphical representations (Battista & Clements, 1996; 1998; Phillips, 1972). Therefore, during the study, both modes of presentations were used to detect if there were any variations in students' functioning.

Through interviews, the strategies used by the students while solving the cube enumeration tasks were obtained in a variety of situations in order to elicit multiple levels of sophistication in solution strategies. Situations included the questions with one-layer, small, medium, and large buildings in both concrete and pictorial representations. The preinterview data were thoroughly examined and partly analyzed before the intervention in order to provide further insight into the instructional intervention phase.

The second phase, intervention, consisted of problems that invited students to share equally the rectangular buildings made of small cubes. Students spent two sessions, approximately 45 minutes each, in this phase. They worked in pairs but the shares varied with respect to the number of columns or layers in buildings. For example, students were asked to share a 3 by 3 by 4 building into either 3 or 4 shares so that they did not have any leftovers after sharing. The purpose of the intervention was twofold:

- One was to see if the students used the same types of conceptualizations as they used while enumerating the cubes in the buildings.

- The other was to see if the students made any improvements in their approaches to the problems during the solutions of sharing activities.

Postinterviews were conducted with each participant six days after the last session of his or her intervention. The reason for these interviews was to probe if any improvement in strategy use was made due to the instructional intervention. The procedural format for the postinterviews was exactly the same as the preinterviews. This phase lasted shorter (three to four minutes for each student because of the mastery) compared to preinterviews (9-11 minutes).

Each phase was videotaped to account for all the actions and verbalizations of the participants while they were attempting to solve the problems with cubes in rectangular buildings both in pictorial and in concrete situations. Retrospective reports (Ericcson & Simon, 1993) were additionally used if the strategy could not be determined through observation by the interviewer. Then the videotapes were transcribed into verbal and graphical (participants' step-by-step coloring and partitioning of the buildings) data in a computer environment. Data were analyzed in an attempt to make sense of the students' structuring of rectangular arrays and the development of this understanding under the circumstances.

Results and Discussion

A qualitative interpretive (Ericksson, 1986) framework was used in the analysis of the data. After multiple readings and examinations of the data generated during the study three main assertions were stated based on the emerging patterns in the data. These assertions were warranted by bringing empirical evidences from the data. Alternative interpretations were also discussed for each assertion.

Assertion 1. *Participants used different strategies depending on the complexity of the building at hand.*

All of the students interviewed enumerated correctly the cubes in Question 1 in both graphical (a 2 x 2 x 1 building) and concrete (a 3 x 3 x1 building) situations. This means that they comprehended both concrete and pictorial representations of one-layer buildings. Since Question 1 (concrete and graphical) had all the cubes visible, there was no ambiguity created on the part of participants. That was why it was eliminated from further analysis; however this knowledge and the level of functioning seemed to be prerequisite for further understanding of rectangular solids made of small cubes.

As shown in Table 1, all the strategies used by the participants for enumerating rectangular buildings made of small cubes fell into categories made by Battista and Clements (1996). (See Appendix A for a detailed description of student strategies). Not one student used Category D or Category E strategies where the student used the formula rotely or misapplied it. This was because the participants of the present study had not yet been introduced to the volume formula.

Table 1. **Participants' Strategies While Enumerating Cubes in Arrays**

Student	**Test**	**Concrete**			**Graphical**		
		2	**3**	**4**	**2**	**3**	**4**
CH	Pre	A2	A2	B2	B2	B2	B2
SA	Pre	A2	A2	C3	A2	B1	C3
ST	Pre	A3	B3	C3	A2	C1	C1
DA	Pre	A2	A3	A3	C1	C1	C1

Note. For the explanation of each strategy, see Appendix A.

The number of unit cubes in the building or the size of the building in three dimensions affected the students' strategy choices. All but one student used Category A strategies for Question 2 (a 2 x 2 x 2 building) in both concrete and pictorial situations in the preinterview (see Table 1). For the fourth question (a 4 x 3 x 3 building), however, two students used Category C strategies, one A, and one B strategy in concrete situations. This difference was even greater in the graphical situation. All but one student used C strategies for Question 4 (pictorial). One student used a B strategy for the same question.

It seemed that when the size of the building increased the task became more complex or overwhelming for the students. If they did not have a consistent strategy, then they used less viable strategies, such as C strategies, in order to find a solution to the problem at hand.

Participants also used different strategies for concrete and pictorial situations although the same-size buildings were used in both of the situations. Students chose Category A strategies more often (8 times in 12) in concrete situations than they did (4 times in 12) in graphical situations. Alternately, they used Category C strategies more often (6 times in 12) in graphical situations than they did (2 times in 12) in concrete situations. This difference was even greater for some students, such as DA and ST, than it was for the other two students.

DA did not make a single mistake nor show any hesitation in determining the number of cubes in one-layer buildings, both concrete and pictorial. He was able to do correctly all the problems presented concretely using Category A strategies; however, he used C1 strategy and did not obtain any correct from the pictorially presented problems. It seemed that he almost completely lacked understanding of three-dimensionality in graphical representations. While finding the number of cubes in pictorial buildings, he consistently counted only the visible cube faces, a typical C1 strategy.

The effect of the size of the building and the effect of pictorial condition were also evident during the intervention. Each time the size of the building was increased, some students, such as DA and ST, seemed overwhelmed, and thus the researcher/teacher had to return to a simpler situation. In addition, students solved the problems with concrete materials more easily than the problems with graphical representations. Therefore, most of the time while working on pictorial representations, the researcher/teacher had to go back to concrete materials, had the students solve the problems and then compare the construction with the pictorial representation. At the very beginning of the intervention, it took the students some time to work with pictorial representations at the same level of ease with which they worked with concrete materials.

This difficulty can be explained by the fact that pictorial situations were perceptually more difficult than concrete situations (Orton & Frobisher, 1996; Cohen, 1972). Additionally, students, at least those participating in this study, were relatively more familiar with concrete materials than they were with pictorial representations of cube buildings. Similar results were reported in the literature (e.g., Battista & Clements, 1996; Ben-Chaim et al., 1985). However, even the experience they had during the intervention was significant in students' acquisition of the skill to accurately interpret two-dimensional graphical representations of certain three-dimensional objects.

Assertion 2*. During the instruction, participants progressed through the three different conceptualizations of rectangular solids composed of small cubes.*

The three conceptualizations are labeled as C, B, and A type for the ease of depiction (see Table 2). Students with a "C" type conceptualization acted on buildings based on faces. They took individual cube faces as unit and overall structuring was based on the building's faces. They did not consider the drawing as three-dimensional and did not take the interior cubes into consideration in the concrete building. Students with "B" conceptualization were aware of the three-dimensionality and space-filling properties of the cubes and the whole building. They used cubes as units but their overall structuring was local, not yet global. For them, the building is a “bunch of cubes.” They usually counted the cubes one by one and unsystematically. It was the "A" conceptualization that enabled students to utilize composite or units of units and unit iteration. For them, the cube building was organized into regular patterns.

Table 2. **Students' Conceptualizations of Rectangular Arrays of Cubes**

Type	Conceptualization	Units formed out of	Overall structuring
C	Cubes as faces	Cube faces	Based on building faces
B	Bunch of cubes	Individual cubes	Local
A	Organized cubes	Cubes, columns, layers	Global

All participants comprehended a one-layer building both in concrete and in pictorial form before the instruction; however, some of them could not mark a layer (i.e., one of the two shares) on the drawing of a two-layer small building. For example, DA and ST were not able to see the equal pieces in the drawings. Instead, they tried to color an equal number of cube faces because their conceptualization of the drawing of rectangular solids was merely a "medley of views" for multilayer buildings. In other words, their initial partitioning was based on two-dimensional faces rather than on three-dimensional spaces.

DA's development through the three types of conceptualizations is depicted in Figure 1. As stated earlier, he completely lacked an understanding of pictorial representations with the exception of one-layered buildings. He first viewed the buildings as consisting of faces –C conceptualization (see Figure 1, view I). Then he vacillated between C and B conceptualization (see Figure 1, view II). His actions while he was coloring the four equal shares were numbered in Figure 1, view II to depict his effort to establish three-dimensional composite units in the picture of a multilayer building. He first colored the front and upper faces of upper-front row (labeled 1) as one of the four shares. He then colored the front face of the lower-front row (labeled 2) as a second share. After coloring the right side of the lower layer (labeled 3), he recognized that the face (labeled 4) belonged to the lower-front row. At his fifth attempt, DA realized that the face (labeled 5) was part of upper-front row. He finally colored the remaining row (labeled 6) without any hesitation.

As seen in Figure 1, view III, he established three-dimensionality in a pictorial representation but could not go beyond local structuring. Although it looks like a layer-type structuring, it is not, because it is supposed to be partitioned into four equal pieces instead of three equal pieces. Finally, towards the end of instruction he reached an "A" conceptualization even with a large building as seen in Figure 1, view IV.

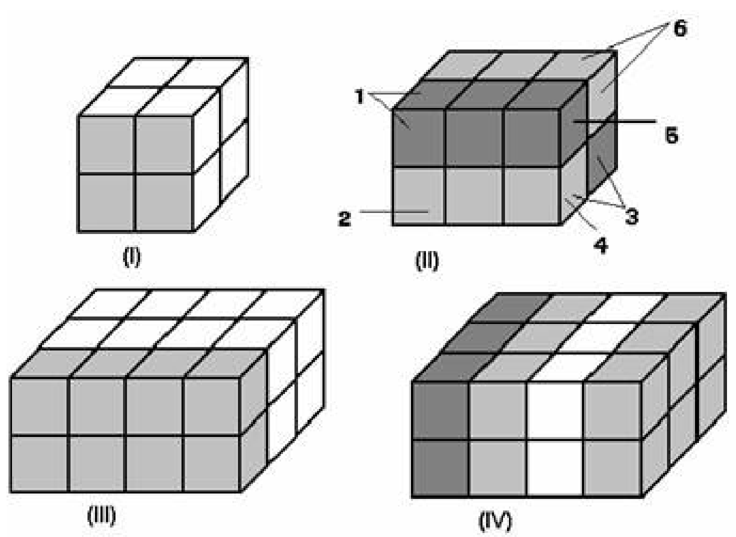

Figure 1. **DA's development through the three types of conceptualizations**

Another student, SA, went through similar stages. During the pre interviews, although she used A2 strategies for enumerating the small buildings, she returned to the C3 strategy for the large building. At the initial stages of the intervention she was not flexible enough to partition the buildings, even the small ones, into a different number of shares. Therefore, she approached the task as "a set of cubes in terms of its faces" (see Figure 2, view I). With encouragement from the interviewer, she established units in the picture based on three dimensions and started to use "local structuring" (see Figure 2, view II). From the small buildings, it was not possible to determine whether her overall structuring was global or local. Both could be possible. However, when the size of the building increased it became obvious that her overall structuring was still local (see Figure 2, views II and III).

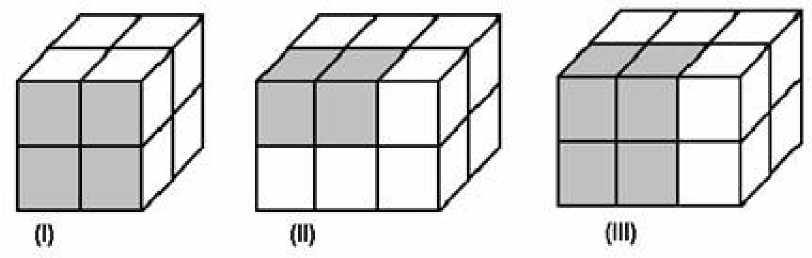

Figure 2. **SA's development from C to B conceptualization for the small and medium buildings**

There was another reason why she used local structuring. She was relying more on a numerical strategy and she was not able to attend to the spatial properties of the whole building. She initially in her mind found the total number of cubes in the building and then divided it by the number of sharers. She then colored that number of cubes. If she obtained the total number correctly, her partitioning was accurate. However, her developing B conceptualization was so fragile that she returned to C conceptualization because of the increasing size of the building (see Figure 3, view I). After awhile, with the help of concrete materials she established the B conceptualization for the large building. However, her partitioning was still not regular (i.e., not based on layers; see Figure 3, view II). Her final strategies towards the end of the instruction were clearly a layering type (see Figure 3, view III). Her actions with the problems that encouraged equal partitioning of the cube buildings helped her to construct the regular spatial patterns in the buildings. SA's three-step progress represented with her drawing tasks is depicted in Figure 3.

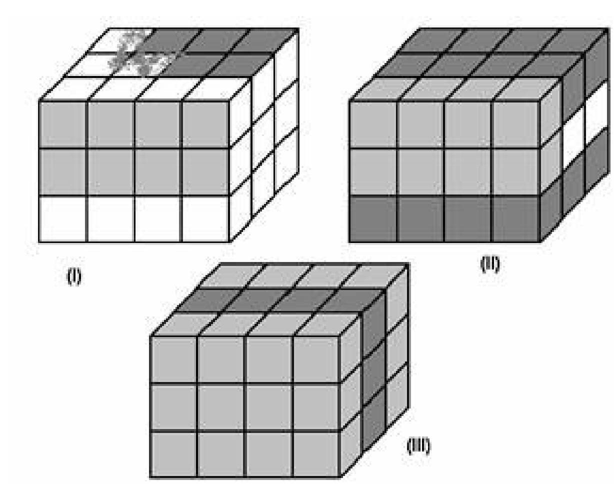

Figure 3. **SA's development through the three types of conceptualizations**

It can be observed from her colorings (see Figure 3) that, again, for the large building SA started with a C conceptualization. Then she managed to move to the B conceptualization. With some guidance from the interviewer and using concrete materials, she was able to use the "A" type conceptualization by systematically and simultaneously acting on both concrete and pictorial representations of the cube arrays. However, even the partitioning of concrete buildings was not easy for the participants at the beginning. They were usually able to construct the equal shares on small concrete buildings. However they were not able right away to do the same with the drawings and the large buildings. These findings further show that drawings of large buildings are especially hard for children to represent mentally.

Assertion 3. *Equal sharing problems paved the road toward more viable strategies.*

The strategies students used in the post interview are depicted in Table 3. All of the participants used Type A strategies after the equal sharing activities.

Table 3. **Participants' Strategies While Enumerating Arrays of Cubes in the Posttest**

Student	**Test**	**Concrete**			**Graphical**		
		2	**3**	**4**	**2**	**3**	**4**
CH	Post	A2	A2	A2	A1	A2	A2
SA	Post	A2	A2	A2	A2	A2	A2
ST	Post	A2	A2	A2	A2	A2	A2
DA	Post	A2	A2	A2	A2	A2	A2

Note. For the explanation of each strategy, see Appendix A.

Such student actions as building rectangular prisms with cubes, the consistent checking of the equality between concrete blocks and their drawings, making equal shares out of prisms, and coloring those shares on pictures brought them to a point where they were able to abstract the equality of some structural elements of the rectangular-cube buildings such as unit cubes, rows, columns, and layers. Gradually, they managed to apply layering strategies to both large size and pictorial buildings.

It is also very possible that the students' specific actions such as coloring the pictures according to concrete blocks, checking for their equality, and iterating equal shares along the third dimension might have helped them to coordinate and integrate different views of unit cubes, composites, and the whole building. By coloring the pictures, students tried to determine the boundaries of unit cubes, columns, and layers while finding equal shares in buildings.

Summary and Conclusions

The results of the present study can be summarized under two subheadings: three different conceptualizations and task complexity. In the first, how students acted on buildings is elaborated. In the second, why students behaved that way is highlighted.

Three different conceptualizations

The participants in this study exhibited and progressed through three different conceptualizations of rectangular solids made of small cubes. This finding is consistent with the spatial structuring theory (Battista & Clements, 1996). Additionally, in each level of conceptualization there were two related processes to be accomplished. One was the formation of units. What the students formed as a unit was inferred from what the students were counting while enumerating the cubes in buildings. They may have counted cube faces, individual cubes, and some composites of cubes.

The other was the overall structuring or organization of these units in the whole building. How the students structured the building was inferred from how they were enumerating the units. The possibilities were counting units with reference to building faces, counting unit cubes in local groups, and counting in regular patterns such as layers.

In the first level, students viewed the buildings as "cubes as faces." Since a typical student at this level could attend to only one face at a time (Battista & Clements, 1996) (i.e., no coordination of views) they took the cube faces as units to be counted and organized them (if any) in terms of building faces. They did not bother with invisible cubes or even faces.

Second, they conceptualized the array as a "bunch of cubes." That means that they are aware of the space-filling and three-dimensionality properties of unit cubes and the building itself but there is no organization of them. Their structuring was based on unit cubes but it was local, not yet global. As a result they tried to count all the cubes inside and out but they could not see any organization of the cubes in the building. It can be said that they coordinated and integrated the different views of a unit cube but not yet the whole building. They first tried to color cubes one by one. They then realized the pattern. At the beginning however, their coloring of equal shares did not correspond to any regular pattern, such as columns and layers of the building. Therefore, their overall structuring was considered local.

Third, students conceptualized the set of cubes as "organized cubes." Students at this level started to properly iterate units based on spatial structural elements of the buildings. They could flexibly view the buildings and form regular patterns such as columns and layers in the building. The global structuring of the whole building was achieved. At the end, skip counting, additive, and multiplicative iterations were successively inserted into their enumeration strategies.

The use of composites depends on the level at which the student performed the coordination and integration operations. While counting, for example, a student may use composites based on two dimensions if he or she did not yet perform any kind of coordination and integration of different views of unit cube and the whole building. Or he or she may use three-dimensional composites if he or she organized, at least partly, the whole building. Thus, a student counting the cubes one by one can be said to be performing coordination and integration operations for individual cubes in a complex building. A student counting the cubes in local groups is additionally trying to coordinate views of the whole building. Finally, a student who use unit or composite unit iterations consistently can be said to be performing the integration operation for the whole building. The proper use of units, composites, and their iterations goes hand in hand with coordination and integration of different views of both unit cubes and then the whole building.

In sum, the difference between the first and the second level conceptualization is the utilization of three-dimensionality or forming proper units based on three dimensions of individual cubes in multilayer buildings. The difference between the second and the third conceptualization is the organization of the structural elements of buildings in a systematic way or forming units of units or composite units based on three-dimensional elements and then unit iteration.

Why is it that the same students used different conceptualizations for different tasks? The main reason for this appears to be the complexity of the buildings such as concrete versus pictorial and small versus large buildings. This will be discussed in the following section.

Task complexity

In preinterviews and instructional periods, students were not consistent in their use of strategies from which their conceptualizations were inferred due to "the task complexity" (Middleton & Corbett, 1998, p. 263). The strategies they used in dealing with cube buildings were distracted by both the increasing size of the buildings and pictorial situations.

It was relatively easier for students to visualize a small building (i.e., a building containing a fewer number of cubes) than a large building. That was why a building increasing in size forced students to use more primitive strategies. Additionally, it was not obvious from the small buildings if the students had a local structuring or a layering-type structuring since the buildings were too small to determine this difference. Later, it became clear from the large buildings that their structuring was local; or it could be that they just started to form the layering-type structuring for small buildings; however, their conceptualization was so fragile that it could easily be distracted by the increasing size of the building.

The pictorial situation was another effect that disturbed the students' spatial structuring. The reason for this appears to be the students' difficulty in visualizing three-dimensionality in pictorial representations. Before the instruction, all the participants used less viable strategies for the problems presented pictorially. In addition, most of the times students used completely different strategies for concrete and pictorial representations.

For students, it was relatively easier to perceive three-dimensionality in concrete representations than it was from pictorial representations. Therefore, students might use a B type strategy for a concrete building, while they were still utilizing a C type strategy for a pictorial representation of the same building. The reason was their difficulty in viewing the three-dimensional units in pictorial representation. Similarly, Battista and Clements (1996) found that students shifted from C to B strategies as they moved from the problems presented pictorially to the ones presented concretely.

Practically, that students can identify an individual cube does not guarantee that they are able to determine cubes in multilayer buildings until they firmly establish the three-dimensional units. Similarly, students may know that they have to find cubes and may determine individual cubes in the building but may not see any organization of them. Only after some experience with appropriate materials (i.e., equal sharing of spatial constructions) can they construct a global structuring of the arrays of cubes.

What specific actions and educational context made this achievement possible for the students? First, students physically and mentally involved in solving the problems provided in equal-sharing context. They made mistakes and corrected themselves. Second, the context required them to construct equal shares out of cubes. Third, the contexts also required them to make the same shares on pictorial representations by coloring the pictures. Through coloring the pictures, they tried to coordinate different views of unit cubes and the whole building. Fourth, after coloring they had a chance to see the pictorial buildings in layers shaded in with different colors. Last, students enjoyed doing all these things.

As a result, it can be said that it is almost meaningless to make the students memorize the volume formula if they do not know anything about the unit of volume and do not see any systematic organization of those units in rectangular prisms. In other words, it is important what they are counting as a unit and how they are enumerating the units in the whole. Without logical support it does not make any sense to them to multiply some numbers. Therefore, emphasis on these kinds of activities is crucial in helping children understand the underlying meaning of the measurement of volume and volume formula.

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Appendix

STUDENTS' STRATEGIES WHILE ENUMERATING CUBES ARRAYS

A. The student conceptualizes the set of cubes as forming a rectangular array organized into layers

1. Layer multiplying: Student computes or counts the number of cubes in one layer (vertical or horizontal) and multiplies by the number of layers.
2. Layer adding/iteration: Student computes the number of cubes in one layer (vertical or horizontal) and uses addition or skip counting (pointing to successive layers) to get the total.
3. Counting subunits of layers: Student's counting of cubes is organized in layers, but the student counts by ones or skip counts by a number that does not equal to the number of cubes in a layer. For example, the student counts the top layer by ones, then counts on from the result, again pointing to each cube in the top layer, for each of the two remaining layers.

B. The student conceptualizes the set of cubes as space filling but does not utilize layers.

1. Column/row iteration: Student counts the number of cubes in one row or column and uses skip counting (pointing to successive rows or columns) to get the total.
2. Counting subunits of columns or rows: Student's counting of cubes is organized by row or column, but the student counts by ones or skip counts by a number that does not equal the number of cubes in a row or column. For example, the student counts by twos or ones, pointing successively to columns of four.
3. Systematic counting: Student counts cubes systematically, attempting to count both inside and outside cubes. He/she might, for instance, count the cubes on all the outside faces, then attempts to determine how many are in the center.*
4. Unsystematic counting: Student counts cubes in a random manner, often omitting or double counting cubes, but clearly tries to account for inside cubes.*

C. The student conceptualizes the set of cubes in terms of its faces.

1. Counting subset of visible cubes: Student counts all, or a subset of, cubes on the front, right side, and top— those that are visible in the picture.*
2. Counting all outside cubes: Student counts outside cubes on all six faces of the prism.*
3. Counting some outside cubes: Student counts outside cubes on some visible and some hidden faces but does not count cubes on all six faces of the prism.*
4. Counting front-layer cubes: Student counts outside cubes in front layer.
5. Counting outside cubes, but not organized by faces.

D. The student uses the formula L x W x H.

Student explicitly says he/she is using formula, or implies it by saying, "Multiply this times this times this" (pointing to relevant dimensions). There is no indication of understanding in terms of layers. (If students used the formula, they were asked, "Why did you multiply these numbers together? Why does this work?")

E. Other.

Student uses a strategy other than those described in A - D, such as multiplying the number of squares on one face times the number on another face.

* This strategy was used, and cubes on some edges were double counted.

Source: Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education, 27*(3), 258 - 292.