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Two anyons with Coulomb interaction in a magnetic field

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We have computed numerically the energy spectrum of two anyons with Coulomb ($1/r$) interaction in a uniform magnetic field. Both non-singular and singular energy eigenfunctions are considered. We solve a second order difference equation with boundary conditions by a new, simple and efficient numerical method.

1. Introduction

The possible existence of two-dimensional particles with fractional statistics, known as anyons, follows from the connection between topology and particle statistics [1–4]. Such particles may exist as quasi-particle excitations in certain two-dimensional systems, and in particular the theory of the fractional quantum Hall effect is based in part on the model of charged anyons in a magnetic field [5,6]. The simple model of non-interacting anyons in a uniform magnetic field is essentially equivalent to that of anyons in a harmonic oscillator potential. The two-particle problem of this kind was solved in ref. [2]. Very recently the lowest energies in the three-particle system have been computed numerically [7,8]. A class of exact many-particle solutions is also known, see for example refs. [9–14].

We present a numerical solution to the problem of two anyons with Coulomb interaction (i.e., a $1/r$ potential) in a uniform magnetic field. Anyons as quasi-particles must have a finite extension in all three spatial dimensions, hence the singularity in the potential is unrealistic, but the simple $1/r$ form makes the numerical solution simple. From the Schrödinger equation we derive in the standard way a second order difference equation for the coefficients of a power series, and we solve this equation numerically, imposing the proper boundary conditions. The special solutions where the power series reduces to a polynomial were found in ref. [15]. Our method is a direct and very efficient numerical method for computing high precision energy eigenvalues in this and in other, similar problems.

Since both the "statistics interaction" and the Coulumb interaction are singular, one should have in mind the possibility of singular wavefunctions, as pointed out in ref. [16]. We discuss here both the standard regular solutions and the singular solutions, which contain an extra continuously variable parameter. The singular wave functions give lower energy levels, and this might indicate that they can be physically realized, although it is not clear what criterion would fix the extra parameter.

Remarkably, all energy eigenstates are bound, even when the Coulomb potential is repulsive. This is probably true, both quantum mechanically and classically, for a quite general radial force between the two particles. Classically, we would expect a drifting motion of the particles around each other, by analogy to the case of one charged particle in uniform, per-

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pendicular electric and magnetic fields, where the time average of the classical motion is a drift perpendicularly to both fields.

2. The eigenvalue equation

The system of interest here consists of two anyons of mass μ and charge q, in a gauge vector potential $A_x=-\frac{1}{2}B_0y$, $A_y=\frac{1}{2}B_0x$. Here B_0 is the constant magnetic flux density, and (x, y) is the position of one particle. We consider only the relative motion, which can be separated from the centre-of-mass motion. The wave function ψ must obey the relation

$$\psi(r,\phi+\pi)=\exp(i\theta)\,\psi(r,\phi)\,, \tag{1}$$

where (r, ϕ) are relative polar coordinates. The angle θ defines the particle statistics.

There exists a complete set of separable energy eigenfunctions of the form

$$\psi(r,\phi)=u(r)\exp(im\phi)\,,$$

$$m=\frac{\theta}{\pi}+2j\,,\quad j=0,\pm1,\pm2,\ldots\,. \tag{2}$$

The radial equation is

$$\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}+2\epsilon m-\rho^2-\frac{b}{\rho}+4\nu\right)u(\rho)=0\,, \tag{3}$$

where (in MKSA units)

$$\rho=\frac{r}{r_0}\,,\quad b=\frac{q^2\mu r_0}{4\pi\epsilon_0\hbar^2}\,,\quad r_0=2\sqrt{\frac{\hbar}{|qB_0|}}\,,$$

$$\nu=\frac{E}{\hbar\omega}\,,\quad \omega=\frac{|qB_0|}{\mu}\,,\quad \epsilon=\mathrm{sign}(qB_0)\,, \tag{4}$$

and E is the energy eigenvalue. Splitting off the asymptotic behaviour at $\rho\to0$ and $\rho\to\infty$ we look for solutions in the form

$$u(\rho)=\rho^{\sigma|m|}\exp(-\tfrac{1}{2}\rho^2)\sum_{k=0}^{\infty}\beta_k\rho^k\,, \tag{5}$$

where $\sigma=\pm1$. The power series is determined up to the normalization factor β_0 by the difference equation

$$(k+2)(k+a)\beta_{k+2}=b\beta_{k+1}+2(k-d)\beta_k\,, \tag{6}$$

where $a=2\sigma|m|+2$ and $d=2\nu-1-\sigma|m|+\epsilon m$. $\sigma=1$ gives a non-singular solution u^{ns}, whereas $\sigma=-1$ gives a singular solution u^{s}. For $|m|\geqslant1$ the singular solution must be rejected because it is non-normalizable, but for $|m|<1$ it cannot be so easily dismissed.

3. The non-singular solutions

Without the Coulomb interaction the odd terms in the power series vanish, and the wave function is normalizable if and only if the power series terminates. In fact, the infinite power series then behaves as $\exp(\rho^2)$ for large ρ, since $\beta_{2k+2}\sim\beta_{2k}/k$ for large k. However, when the even and odd terms are coupled by the Coulomb term in eq. (6), there may exist infinite series solutions with physically acceptable behaviour.

To investigate this further we write $\beta_k=T_k\gamma_k$ with

$$T_k=\frac{\Gamma(\frac{1}{2}(k-d))}{\Gamma(\frac{1}{2}(k+2))\Gamma(\frac{1}{2}(k+a))}\,, \tag{7}$$

and obtain the equation

$$\gamma_{k+1}=b_k\gamma_k+\gamma_{k-1}\,,$$

$$b_k=\frac{b}{4}\frac{\Gamma(\frac{1}{2}(k+1))\Gamma(\frac{1}{2}(k+a-1))\Gamma(\frac{1}{2}(k-d))}{\Gamma(\frac{1}{2}(k+2))\Gamma(\frac{1}{2}(k+a))\Gamma(\frac{1}{2}(k-d+1))}\,. \tag{8}$$

For large k we have that $b_k\sim b/\sqrt{2k^3}$. Assume that $k\geqslant K$, for some K, and let M be the maximum of $|\gamma_K|$ and $|\gamma_{K+1}|$. Then we have from eq. (8) that

$$|\gamma_k|\leqslant M\prod_{i=K+1}^{k-1}(1+|b_i|)\leqslant M\prod_{i=K+1}^{\infty}(1+|b_i|)$$

$$=M'\,. \tag{9}$$

The infinite product converges because $\sum|b_i|$ converges, so $|\gamma_k|$ is bounded by the constant M'. Furthermore, when $n>m$

$$|\gamma_{2n-1}-\gamma_{2m-1}|=\left|\sum_{i=m}^{n-1}b_{2i}\gamma_{2i}\right|\leqslant M'\sum_{i=m}^{\infty}|b_{2i}|\,, \tag{10}$$

$$|\gamma_{2n}-\gamma_{2m}|=\left|\sum_{i=m}^{n-1}b_{2i+1}\gamma_{2i+1}\right|\leqslant M'\sum_{i=m}^{\infty}|b_{2i+1}|\,. \tag{11}$$

This shows that the odd and even sequences are Cauchy sequences so that the limits

$$\gamma_\infty^{\text{even}} = \lim_{m\to\infty} \gamma_{2m}\,, \quad \gamma_\infty^{\text{odd}} = \lim_{m\to\infty} \gamma_{2m+1} \tag{12}$$

exist. To prove that these two limits may be chosen as the two free parameters in the general solution of the difference equation, we write eq. (8) in matrix form, $Y_{k+1}=B_kY_k$, with

$$Y_k=\begin{pmatrix}\gamma_{2k}\\ \gamma_{2k-1}\end{pmatrix},$$

$$B_k=\begin{pmatrix}b_{2k+1} & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}b_{2k} & 1\\ 1 & 0\end{pmatrix}. \tag{13}$$

The transfer matrix $B_{k\to m}=B_{m-1}\ldots B_{k+1}B_k$ has determinant 1, hence $Y_k=(B_{k\to\infty})^{-1}Y_\infty$. It follows that there exist two linearly independent solutions

$$\beta_k^+ = T_k(1+\delta_k^+)\,, \quad \beta_k^- = (-1)^k T_k(1+\delta_k^-)\,, \tag{14}$$

with $\delta_k^\pm \to 0$ as $k\to\infty$. They make $|\beta_k|$ the smoothest possible function of k. The positive solution β_k^+ gives a power series dominating $\exp(\rho^2)$ and must be rejected. The alternating solution β_k^- gives maximal cancellation between successive terms in the power series, and is the physically acceptable solution.

To find either one of the solutions $\beta_k^\pm$ numerically, we use eq. (6), dividing by β_{k+1} to get a recursion relation for the ratios $R_k=\beta_{k+1}/\beta_k$. We start at a suitable value $k=K$, say $K=50$ or 200. The equation $R_{K+1}=R_K$ gives a first approximation to R_K. We choose the unique positive root to get β_k^+, or the unique negative root to get β_k^-. To improve the approximation we define the ratios $R_k^{(n+1)}=R_{k+1}^{(n)}/R_k^{(n)}$, with $R_k^{(0)}=R_k$. By requiring that $R_k^{(n)}=1$ for various n we obtain various degrees of smoothing. We solve the equation $R_k^{(n)}=1$ iteratively by multiplying (if n is odd) or dividing (if n is even) the present approximate value of R_K by $\sqrt[N]{R_k^{(n)}}$, where $N=2^n$. The numerical proof that we get the correct asymptotic solution is that R_k for a given k is unchanged if we increase K. The sequence β_k we get by iterating eq. (6) upward from $k=0$ has to match the asymptotic solution β_k^-, and this condition determines the energy eigenvalues. Special solutions are those where the power series terminates.

Fig. 1 and fig. 2 show examples of the m and b dependence of the lowest energy levels. Second order perturbation theory is good below $|b|\simeq 1$, in spite of the singularity of the Coulomb potential, but clearly does not work for realistic values of $b\approx 10^2$–10^3.

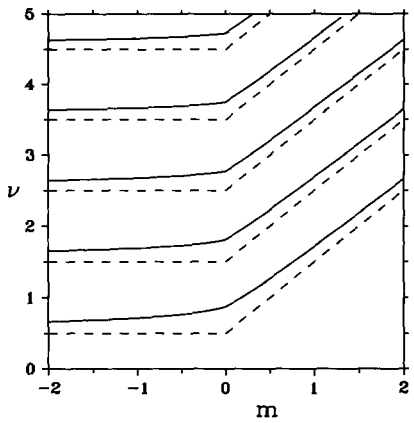

Fig. 1. Energy (in units of $\hbar\omega$) versus angular momentum m for non-singular wave functions. We take $qB_0<0$. Solid lines: $b=1$; dashed lines: $b=0$.

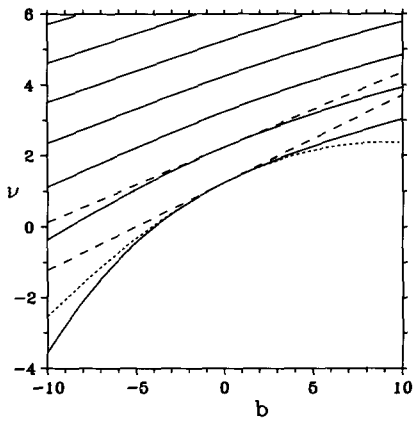

Fig. 2. Energy versus Coulomb coupling b for non-singular wave functions. $m=0.75$, $qB_0<0$. Dashed lines: first order perturbation theory for the two lowest levels. Dotted line: second order ground state energy.

4. The singular solutions

For $|m|<1$ one may accept solutions of the form $m=u^{\text{ns}}+u^{\text{s}}$, but restricted by the condition that the ratio $\lambda=\beta_0^{\text{s}}/\beta_0^{\text{ns}}$ must have a fixed real value. Note that for any given statistics angle $\theta\neq\pm\pi, \pm 3\pi, \ldots$ there

is exactly one value of m with $|m|<1$. The corresponding λ is a second free parameter defining the system, in addition to θ [16]. $\lambda=0$ gives the non-singular solution u^{ns}, and $1/\lambda=0$ gives the particular singular solution u^{s}.

The two solutions u^{ns} and u^{s} can be computed numerically in the way already described. For other values of λ we must compute the two sequences β_k^{ns} and β_k^{s}, using eq. (6) with the initial values $\beta_0^{ns}=1$, $\beta_0^{s}=\lambda$, and write them as linear combinations of the asymptotic solutions $\beta_k^{\pm}$,

$$\beta_k^{ns}=\alpha_+^{ns}\beta_k^{ns+}+\alpha_-^{ns}\beta_k^{ns-},$$

$$\beta_k^{s}=\alpha_+^{s}\beta_k^{s+}+\alpha_-^{s}\beta_k^{s-}. \qquad (15)$$

The energy eigenvalue condition is that the terms

$$\sum_k (\alpha_+^{ns}\beta_k^{ns+}\rho^{k+|m|}+\alpha_+^{s}\beta_k^{s+}\rho^{k-|m|}) \qquad (16)$$

in the expansion for $u(\rho)$ should cancel maximally. This happens when

$$\alpha_+^{ns}\beta_k^{ns+}=f(k+|m|),$$

$$\alpha_+^{s}\beta_k^{s+}=-f(k-|m|), \qquad (17)$$

where f is some smooth function.

Note that u^{s} coicides with u^{ns} as either $m\to 0$, $|m|\to\frac{1}{2}$ or $|m|\to 1$, and the solutions that go to singular limits are mixtures of u^{s} and u^{ns}, with $\lambda\to -1$ as $m\to 0$ and $\lambda\to 0$ as $|m|\to\frac{1}{2}$ or $|m|\to 1$. Note also that the expectation value of the Coulomb potential in a singular state is infinite when $|m|\geq\frac{1}{2}$. Therefore the perturbation expansion around $b=0$ breaks down, and in an energy eigenstate the kinetic energy part of the hamiltonian must have expectation value $+\infty$ if $b<0$ or $-\infty$ (!!) if $b>0$.

Fig. 3 shows the m dependence of the lowest energy levels when λ is chosen in a particular, rather arbitrary way. For each m we choose λ so as to get one energy level midway between the two lowest energy levels with non-singular wave functions. The figure shows also how λ varies with m. Note that it is impossible for more than one single energy eigenfunction to remain singular in the limit $|m|\to 1$.

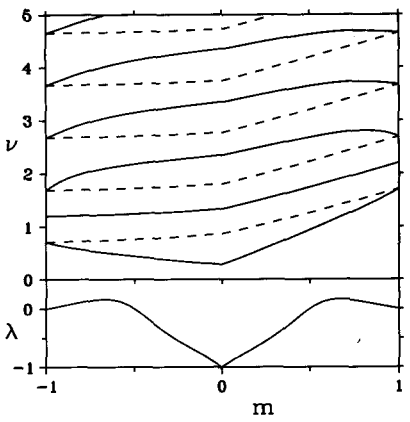

Fig. 3. Energy versus m for singular wave functions (see text). $b=1$, $qB_0<0$. Dashed lines: energy levels from fig. 1. λ versus m is plotted below.

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