



ON SURFACES WITH COMMON PSEUDO NULL GEODESIC IN MINKOWSKI 3-SPACE

UFUK OZTURK

ABSTRACT. The problem of constructing a family of parametric surfaces from a given pseudo null curve using the Frenet trihedron frame of the curve in the Minkowski 3-space is studied. The necessary and sufficient conditions for the given pseudo null curve to be a common geodesic of the parametric surface are derived. As a member of the family, considered a ruled surface with a pseudo null directrix, and some corollaries on the ruled surface are given. Also, some examples are given to show the family of surfaces with a common pseudo null geodesic.

1. INTRODUCTION

At the beginning of the twentieth century, A. Einstein's theory opened a door for use of new geometries. One of them is the geometry of special relativity, which induced on each fixed tangent space of an arbitrary Lorentzian manifold. Some of classical differential geometry topics have been treated by the researchers.

In the Euclidean plane, a straight line is the shortest path between any two points. But in Euclidean 3-space, a geodesic is the shortest route between two points, so geodesics play the role of straight lines. Therefore, geodesic curves or geodesics play a crucial role in many fields such as computer aided geometric design and robotics etc. (see [4, 6, 9]).

In Euclidean 3-space, proposed a principle for constructing a surface pencil with a common isogeodesic curve and focused on the reverse problem by Wang, Tang, and Tai in [11]: given a space curve, how to characterize those surfaces that possess this curve as a common geodesic . They also derived a sufficient condition on marching-scale functions for which the curve is an isogeodesic curve on a given surface with the geodesic and isoparametric constraints. In particular, it is generalized the work of Wang and introduced new types coefficients for a given curve on the parametric surface by Kasap, Akyildiz and Orbay in [3]. In Minkowski 3-space,

Received by the editors: June 20, 2016, Accepted: November 15, 2016.

2010 *Mathematics Subject Classification.* 53C22, 53C40.

Key words and phrases. Pseudo null curve, Geodesic curve, Parametric surface, Minkowski 3-space.

surfaces with a common geodesic are studied and the necessary and sufficient conditions on marching-scale functions are given by Kasap and Akyildiz in [2]. Lately, the family of surfaces with a common null geodesic is studied by Safak and Kasap in [8].

In this paper, the problem on how to construct a surface family from a given pseudo null curve, which is a spacelike curve with a null principal normal, is studied in Minkowski 3-space \mathbb{E}_1^3 . In Section 2, the basic definitions of Minkowski 3-space \mathbb{E}_1^3 are explained that will be used throughout the paper. In Section 3, it is shown that how to derive the necessary and sufficient conditions for the given pseudo null curve to be an isoparametric and a geodesic for the parametric surface. By the way, some representative curves are given to construct the corresponding surfaces which possess these curves as geodesic curves. In Section 4, when a ruled surface is considered as a member of the family, some corollaries and examples are given to illustrate for the ruled surface.

2. PRELIMINARIES

The Minkowski 3-space \mathbb{E}_1^3 is the Euclidean 3-space \mathbb{E}^3 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2, \quad (2.1)$$

where (x_1, x_2, x_3) is a rectangular coordinate system of \mathbb{E}_1^3 . Since g is an indefinite metric, recall that a vector $v \in \mathbb{E}_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$ and $v \neq 0$. In particular, the norm (length) of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$ and two vectors \vec{v} and \vec{w} are said to be orthogonal, if $g(v, w) = 0$. Next, recall that an arbitrary curve $\alpha = \alpha(s)$ in \mathbb{E}_1^3 , can locally be *spacelike*, *timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are respectively *spacelike*, *timelike* or *null (lightlike)* (for more details see [5]).

If α is a pseudo null curve which is a spacelike curve with a null principal normal N , the Frenet formulas have the form (see [7, 10])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (2.2)$$

where

$$\begin{aligned} T(s) &= \alpha'(s), \\ N(s) &= \alpha''(s), \end{aligned}$$

and

$$\begin{aligned} g(T, T) &= 1, \quad g(N, N) = g(B, B) = 0, \\ g(T, N) &= g(T, B) = 0, \quad g(N, B) = 1. \end{aligned} \quad (2.3)$$

For a pseudo null curve, the first curvature κ can take only two values: $\kappa = 0$ when α is a straight line, or $\kappa = 1$ in all other cases.

The Lorentzian vector product of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in \mathbb{E}_1^3 is defined by

$$\begin{aligned} u \times v &= \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1). \end{aligned}$$

Lemma 1. *Let u, v and w be vectors in \mathbb{E}_1^3 . Then:*

- (i) $g(u \times v, w) = \det(u, v, w),$
- (ii) $u \times (v \times w) = -g(u, w)v + g(u, v)w,$
- (iii) $g(u \times v, u \times v) = -g(u, u)g(v, v) + g(u, v)^2.$

Assume that $\vec{\alpha}(s)$ is a unit speed pseudo null curve whose Frenet frame field is $\{T(s), N(s), B(s)\}$ in \mathbb{E}_1^3 . Then, we have

$$T \times N = N, \quad N \times B = T, \quad B \times T = B. \tag{2.4}$$

In this paper, we will assume that pseudo null base curve α has the first curvature $\kappa(s) = 1$, i.e. that the curve α is not a straight line.

3. SURFACES WITH A COMMON PSEUDO NULL GEODESIC CURVE IN \mathbb{E}_1^3

Let S be a parametric surface which is given with a pseudo null curve $\alpha = \alpha(s)$ in the 3-dimensional Minkowski space. Then,

$$\begin{aligned} \phi : [0, L] \times [0, T] &\rightarrow \mathbb{E}_1^3 \\ (s, t) &\longrightarrow \phi(s, t) = \alpha(s) + \mathbf{e}(s, t), \end{aligned} \tag{3.1}$$

where $a(s, t), b(s, t)$ and $c(s, t)$ are C^1 functions and \mathbf{e} is given by

$$\mathbf{e}(s, t) = a(s, t)T(s) + b(s, t)N(s) + c(s, t)B(s). \tag{3.2}$$

If the parameter t is seen as the time, then the functions $a(s, t), b(s, t)$ and $c(s, t)$ can be viewed as directed marching distances of a point unit in the time t in the direction T, N and B , respectively, and the position vector α is seen as the initial location of this point.

By taking the derivative of the equation (3.2) with respect to s and using the Frenet equations (2.2), we get

$$\begin{aligned} \mathbf{e}_s &= \frac{\partial \mathbf{e}}{\partial s} \\ &= (a_s - c)T + (b_s + b\tau + a)N + (c_s - c\tau)B. \end{aligned} \tag{3.3}$$

The normal $\eta(s, t)$ of the surface S is given by

$$\eta(s, t) = \phi_s \times \phi_t,$$

and since

$$\phi_s = \frac{\partial \phi}{\partial s} = (a_s - c + 1)T + (b_s + b\tau + a)N + (c_s - c\tau)B,$$

and

$$\phi_t = \frac{\partial \phi}{\partial t} = a_t T + b_t N + c_t B,$$

using the equations (2.4), the normal vector can be written as

$$\begin{aligned} \eta(s, t) &= ((b_s + b\tau + a)c_t - (c_s - c\tau)b_t)T \\ &+ ((a_s - c + 1)b_t - (b_s + b\tau + a)a_t)N \\ &+ ((c_s - c\tau)a_t - (a_s - c + 1)c_t)B. \end{aligned} \quad (3.4)$$

Let the curve $\alpha = \alpha(s)$ on the surface S , given by equation (3.1), be an isoparametric. Then, there should exist a parameter $t = t_0$ such that $\alpha(s) = \phi(s, t_0)$ where

$$a(s, t_0) = b(s, t_0) = c(s, t_0) = 0, \quad (3.5)$$

and from (3.4) we obtain

$$\eta(s, t_0) = (b_s c_t - c_s b_t)T + ((a_s + 1)b_t - b_s a_t)N + (c_s a_t - (a_s + 1)c_t)B.$$

According to the geodesic theory in [1], the curve α on the surface S is a geodesic if only if the principal normal $N(s)$ of the curve α and the normal $\eta(s, t_0)$ of the surface S at any point on the curve α are parallel to each other. Thus, for all $s \in [0, L]$

$$N(s) \parallel \eta(s, t_0)$$

if and only if

$$\begin{aligned} b_s c_t - c_s b_t &= 0, \\ (a_s + 1)b_t - b_s a_t &\neq 0, \\ c_s a_t - (a_s + 1)c_t &= 0. \end{aligned} \quad (3.6)$$

Therefore, we can give the necessary and sufficient conditions for the surface S to have the pseudo null curve α as an isoparametric and a geodesic with the following theorem.

Theorem 1. *Let S be a parametric surface given by (3.1) in Minkowski 3-space. The curve α is an isoparametric and a geodesic curve on the surface S if and only if the following conditions are satisfied*

$$\begin{aligned} a(s, t_0) = b(s, t_0) = c(s, t_0) &= 0, \\ b_s c_t - c_s b_t &= 0, \\ (a_s + 1)b_t - b_s a_t &\neq 0, \\ c_s a_t - (a_s + 1)c_t &= 0, \end{aligned}$$

where $a(s, t)$, $b(s, t)$ and $c(s, t)$ are C^1 functions.

We call the set of surfaces defined by the equations (3.1), (3.5) and (3.6) a *family of surfaces with a common pseudo null geodesic*, where the common pseudo null geodesic is both an isoparametric and a geodesic curve on these surfaces. Any surface S is defined by the equation (3.1) and satisfying for the equations (3.5) and (3.6) is a member of the family.

As mentioned in [11], the marching-scale functions $a(s, t)$, $b(s, t)$ and $c(s, t)$ can be decomposed into two factors:

Case 1. *If we choose*

$$\begin{aligned} a(s, t) &= l(s)A(t), \\ b(s, t) &= m(s)B(t), \\ c(s, t) &= n(s)C(t), \end{aligned} \tag{3.7}$$

where $l(s)$, $m(s)$, $n(s)$, $A(t)$, $B(t)$ and $C(t)$ are C^1 functions, and $l(s)$, $m(s)$ and $n(s)$ are not identically zero. Then, from Theorem 1 we can simply express the necessary and sufficient conditions for the pseudo null curve α being an isoparametric and geodesic curve on the surface S as follows

$$\begin{cases} A(t_0) = B(t_0) = C(t_0) = 0, \\ \frac{dB(t_0)}{dt} = \text{const} \neq 0, \quad m(s) \neq 0, \\ \frac{dC(t_0)}{dt} = 0, \text{ or } n(s) = 0, \end{cases} \tag{3.8}$$

When the marching-scale functions $a(s, t)$, $b(s, t)$ and $c(s, t)$ depend only on the parameter t , if we choose $l(s) = m(s) = n(s) = 1$, then the corresponding family of surfaces with a common pseudo null geodesic becomes

$$\phi(s, t) = \alpha(s) + A(t)T(s) + B(t)N(s) + C(t)B(s).$$

By simplifying, condition (3.8) can be represented as

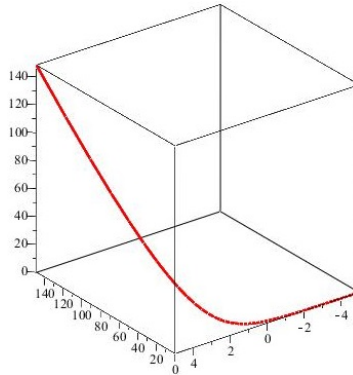
$$\begin{aligned} A(t_0) = B(t_0) = C(t_0) &= 0, \\ \frac{dB(t_0)}{dt} = \text{const} \neq 0, \quad \frac{dC(t_0)}{dt} &= 0. \end{aligned}$$

Case 2. *If we choose*

$$\begin{aligned} a(s, t) &= f(l(s)A(t)), \\ b(s, t) &= g(m(s)B(t)), \\ c(s, t) &= h(n(s)C(t)), \end{aligned}$$

where $l(s)$, $m(s)$, $n(s)$, $A(t)$, $B(t)$, $C(t)$, f , g , and h are C^1 functions. Then, from Theorem 1 and (3.8) we can simply express the necessary and sufficient conditions for the pseudo null curve α being an isoparametric and geodesic curve on the surface S as follows

$$\begin{cases} A(t_0) = B(t_0) = C(t_0) = 0, \\ f(0) = g(0) = h(0) = 0, \\ \frac{dB(t_0)}{dt} = \text{const} \neq 0, \quad m(s) \neq 0, \quad g'(0) \neq 0, \\ \frac{dC(t_0)}{dt} = 0, \text{ or } n(s) = 0, \text{ or } h'(0) = 0, \end{cases} \tag{3.9}$$

FIGURE 1. The curve α

The choices above give an advantage: any set of functions $l(s)$, $m(s)$, and $n(s)$ would satisfy (3.8) or (3.9). Thus we can select different sets of functions $l(s)$, $m(s)$, and $n(s)$ to adjust the shape of the surface until they are gratified with the design, and the resulting surface is guaranteed to belong to the surface family with the pseudo null curve $\alpha(s)$ as the common geodesic.

Example 1. Let S be a parametric surface given by (3.1) in Minkowski 3-space. Let us consider the pseudo null helix with parameter equation (see Fig. 1)

$$\alpha(s) = (\sinh s + \cosh s, s, \sinh s + \cosh s).$$

Then we get the Frenet vectors as follows:

$$\begin{aligned} \vec{T}(s) &= \alpha'(s) = (\sinh s + \cosh s, 1, \sinh s + \cosh s), \\ \vec{N}(s) &= \alpha''(s) = (\sinh s + \cosh s, 0, \sinh s + \cosh s), \\ \vec{B}(s) &= \left(-\frac{(\sinh s + \cosh s)^2 + 1}{2(\sinh s + \cosh s)}, -1, \frac{1 - (\sinh s + \cosh s)^2}{2(\sinh s + \cosh s)} \right). \end{aligned}$$

Moreover, the curvatures κ and τ of α have the form

$$\kappa(s) = 1 \text{ and } \tau(s) = 1.$$

If we choose

$$\begin{aligned} a(s, t) &= \gamma \sin t, \\ b(s, t) &= \delta t, \\ c(s, t) &= 0, \end{aligned}$$

where $\gamma, \delta \in \mathbb{R}$ and $\delta \neq 0$. Then, the surface family with the common pseudo null geodesic is given by

$$\phi(s, t) = \begin{pmatrix} (\cosh s + \sinh s)(\gamma \sin t + \delta t + 1), \gamma \sin t + s, \\ (\cosh s + \sinh s)(\gamma \sin t + \delta t + 1) \end{pmatrix}.$$

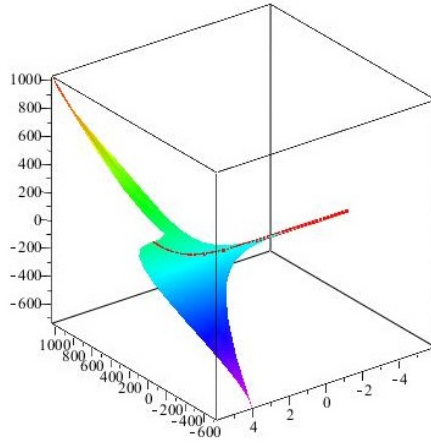


FIGURE 2. A member of surface family for $\gamma = 1, \delta = -1$ and the curve α

For $\gamma = 1$ and $\delta = -1$ we obtain a member of the surface (see Fig. 2) as

$$\phi(s, t) = \begin{pmatrix} (\cosh s + \sinh s) (\sin t - t + 1), \sin t + s, \\ (\cosh s + \sinh s) (\sin t - t + 1) \end{pmatrix},$$

where $t \in [-5, 5]$.

For $\gamma = \sin t$ and $\delta = 1$ we obtain a member of the surface (see Fig. 3) as

$$\phi(s, t) = \begin{pmatrix} (\cosh s + \sinh s) (\sin^2 t + t + 1), \sin^2 t + s, \\ (\cosh s + \sinh s) (\sin^2 t + t + 1) \end{pmatrix},$$

where $t \in [-5, 5]$.

4. RULED SURFACE WITH A COMMON PSEUDO NULL GEODESIC CURVE

For a member of the family is defined by (3.1), consider a ruled surface $\phi(s, t)$ with a pseudo null directrix $\alpha(s)$ which is also an isoparametric. In that case, there exists a parameter $t = t_0$ such that $\phi(s, t_0) = \alpha(s)$. Then, for $s \in [0, L]$ and $t_0, t \in [0, T]$ the surface $\phi(s, t)$ can be expressed as

$$\phi(s, t) - \phi(s, t_0) = (t - t_0) D(s),$$

where $D(s)$ denotes the direction of the rulings.

Also, from the equations (3.1) and (3.2), we get

$$(t - t_0) D(s) = a(s, t)T(s) + b(s, t)N(s) + c(s, t)B(s).$$

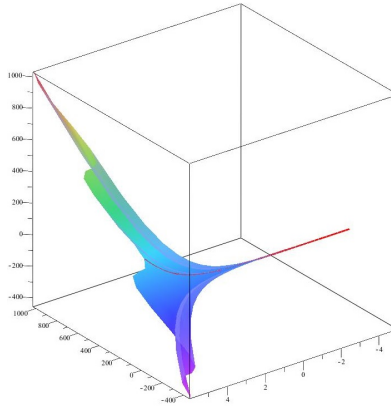


FIGURE 3. A member of surface family for $\gamma = \sin t$, $\delta = 1$ and the curve α

Using the equations (2.3) and (2.4), a system of three equations with three unknown functions $a(s, t)$, $b(s, t)$ and $c(s, t)$ can be written as

$$\begin{aligned} a(s, t) &= (t - t_0) \det [D(s), N(s), B(s)], \\ b(s, t) &= (t - t_0) \det [D(s), B(s), T(s)], \\ c(s, t) &= (t - t_0) \det [D(s), T(s), N(s)]. \end{aligned} \quad (4.1)$$

The above equations in (4.1) are just the necessary and sufficient conditions for a ruled surface $\phi(s, t)$ to have a pseudo null directrix $\alpha(s)$ as an isoparametric.

If the curve $\alpha(s)$ is also a geodesic on the surface $\phi(s, t)$ by using the conditions given in (3.8), then we get

$$\begin{aligned} \det [D(s), B(s), T(s)] &\neq 0, \\ \det [D(s), T(s), N(s)] &= 0. \end{aligned} \quad (4.2)$$

It follows that at any point on the curve $\alpha(s)$, the ruling direction $D(s)$ must be in the plane spanned by $T(s)$ and $N(s)$. Moreover, the ruling direction $D(s)$ and the vector $T(s)$ must not be parallel. Thus, for some real functions $u(s)$ and $v(s)$, we can write

$$D(s) = u(s)T(s) + v(s)N(s), \quad (4.3)$$

where $v(s) \neq 0$ for all $s \in [0, L]$. Substituting it in (4.1)-(4.3), we have

$$\begin{aligned} a(s, t) &= tu(s), \\ b(s, t) &= tv(s), \\ c(s, t) &= 0, \end{aligned} \quad (4.4)$$

where $b(s, t) \neq 0$ for all $s \in [0, L]$.

Thus, the ruled surface with the common pseudo null geodesic directrix $\alpha(s)$ is given by

$$\phi(s, t) = \alpha(s) + t(u(s)T(s) + v(s)N(s)), \quad (4.5)$$

where the real functions $u(s)$ and $v(s)$ control the shape of the ruled surface, and $v(s) \neq 0$ for all $s \in [0, L]$. On the other hand, there exist two geodesic curves passing through every point on the curve $\alpha(s)$: one is $\alpha(s)$ itself and the other is a straight line in the direction $D(s)$ as given in (4.3). Every member of the ruled surface with the common pseudo null geodesic directrix can be expressed by the direction vector function $D(s)$.

Case 3. *If the ruled surface, which is given by (4.5), is a cylindrical surface, then we can write*

$$D(s) \times D'(s) = 0, \quad (4.6)$$

for all $s \in [0, L]$. So, there exists a function $\mu(s)$ such that

$$D'(s) = \mu(s) D(s), \quad (4.7)$$

for all $s \in [0, L]$. From (2.2) and (4.3) we get the system as follows:

$$u'(s) = \mu(s) u(s), \quad (4.8)$$

and

$$v'(s) + u(s) + v(s)\tau(s) = \mu(s) v(s). \quad (4.9)$$

The general solution of (4.8) is given by

$$u(s) = C_1 e^{\int \mu(s) ds}, \quad (4.10)$$

where C_1 is a constant of integration. From (4.9) and (4.10), we obtain

$$v(s) = \left(C_2 + C_1 \int e^{\int \tau(s) ds} ds \right) e^{\int (\tau(s) - \mu(s)) ds}, \quad (4.11)$$

where C_1 and C_2 are constants of integration.

If $\tau(s) = 0$ for all $s \in [0, L]$, then the last equation becomes

$$v(s) = (C_1 s + C_2) e^{-\int \mu(s) ds}. \quad (4.12)$$

Therefore, we can give the following corollary.

Corollary 1. *Let the ruled surface, which is given by (4.5), be a cylindrical surface.*

If the ruled surface has two of the following properties, it has the third also,

- (i) *The base curve is a planar curve,*
- (ii) *The base curve is both a geodesic and an isoparametric,*
- (iii) *The real functions $u(s)$ and $v(s)$ are given by*

$$\begin{aligned} u(s) &= C_1 e^{\int \mu(s) ds}, \\ v(s) &= (C_1 s + C_2) e^{-\int \mu(s) ds}, \end{aligned}$$

where C_1 and C_2 are constants of integration and $v(s) \neq 0$ for all $s \in [0, L]$.

Case 4. If the ruled surface, which is given by (4.5), is a non-cylindrical surface, then we can write

$$D(s) \times D'(s) \neq 0, \tag{4.13}$$

for all $s \in [0, L]$. Also, from (2.2)-(2.4) and (4.3) we have

$$g(D(s) \times D'(s), \alpha'(s)) = \det [D(s), D'(s), \alpha'(s)] = 0, \tag{4.14}$$

for all $s \in [0, L]$.

Thus, we can give the following corollary.

Corollary 2. If the ruled surface, which is given by (4.5), is a non-cylindrical surface, then the ruled surface is always a developable surface for all $s \in [0, L]$ and $t \in [0, T]$.

Example 2. Let S be a ruled surface whose geodesic curve is the pseudo null helix in Example 1, with $C_1 = 1$, $C_2 = 1$ and $\mu(s) = 1$ for all s .

If the controlling functions of the ruled surface for all s are $u(s) = e^s$ and $v(s) = 1 + e^s$, then the corresponding cylindrical surface is drawn as in Figure 4.

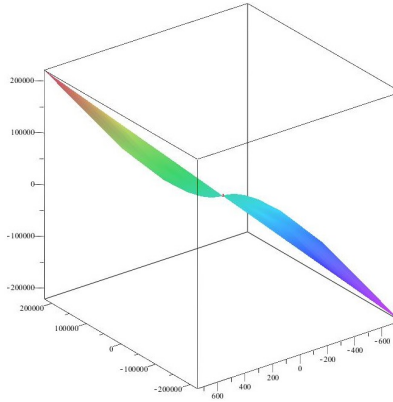
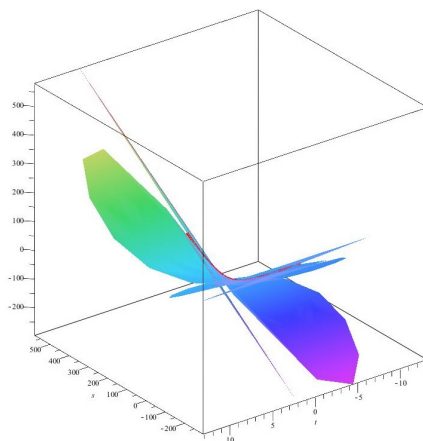
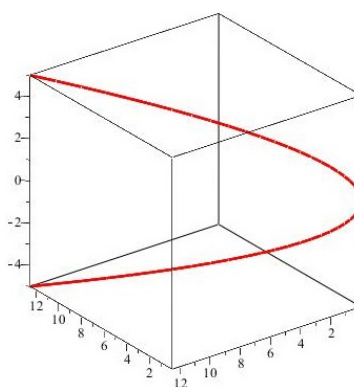


FIGURE 4. The ruled surface ϕ with pseudo null helix

If the controlling functions of the ruled surface for all s are $u(s) = \cos s$ and $v(s) = e^{-s}$, then the corresponding non-cylindrical surface is shown in Figure 5.

Example 3. Let S be a ruled surface with a pseudo null directrix α in \mathbb{E}_1^3 . Let us consider the pseudo null helix with parameter equation (see Fig. 6)

$$\alpha(s) = \left(\frac{s^2}{2}, \frac{s^2}{2}, s \right). \tag{4.15}$$

FIGURE 5. The ruled surface ϕ with pseudo null helixFIGURE 6. The curve α

Then we get the Frenet vectors as follows

$$\begin{aligned}\vec{T}(s) &= \alpha'(s) = (s, s, 1), \\ \vec{N}(s) &= \alpha''(s) = (1, 1, 0), \\ \vec{B}(s) &= \left(\frac{1+s^2}{2}, \frac{s^2-1}{2}, s\right).\end{aligned}\tag{4.16}$$

Moreover, the curvatures κ and τ of α have the form

$$\kappa(s) = 1 \text{ and } \tau(s) = 0.\tag{4.17}$$

If the controlling functions of the ruled surface for $C_1 = 1$, $C_2 = 1$ and $\mu(s) = 0$ for all s are

$$\begin{aligned} u(s) &= 1, \\ v(s) &= 1 + s, \end{aligned} \quad (4.18)$$

then the corresponding cylindrical surface is drawn as in Figure 7.

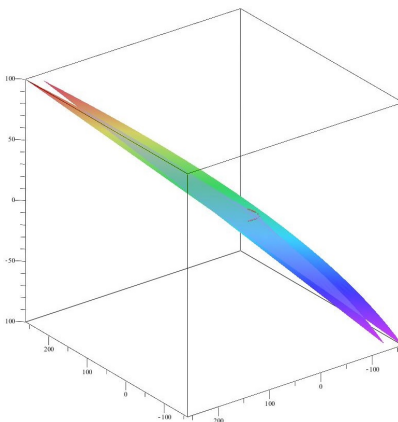


FIGURE 7. The curve α

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Current address: Department of Mathematics, Çankırı Karatekin University, Çankırı, 18100, Turkey

E-mail address: ozturkufuk06@gmail.com, uuzturk@asu.edu