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ON PROXIMATE ORDER AND PROXIMATE TYPE OF ENTIRE DIRICHLET SERIES

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Abstract. In this paper we introduce the notion of Proximate Order and Proximate Type of Entire Dirichlet Series and prove their existence. We also obtain some related results.

1. Introduction

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined by the everywhere convergent Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1}$$

where $0 < \lambda_n < \lambda_{n+1}(n \geq 1), \lambda_n \to \infty \; as \; n \to \infty$ and $a_n \in \mathbb{C}$.

If σ_c and σ_a be respectively the abscissa of convergence and absolute convergence of (1) then $\sigma_c = \sigma_a = \infty$.

For an entire function $f(s)$ represented by (1) its maximum modulus is denoted by $F(\sigma)$ and is defined as

$$F(\sigma) = \sup \{|f(\sigma + it)| : t \in \mathbb{R}\}.$$

The Ritt order ρ_f of $f(s)$ is defined as

$$\rho_f = \limsup_{\sigma \to \infty} \frac{\log^{[2]} F(\sigma)}{\sigma}$$

where (cf. [5]):

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3 \ldots \text{ and } \log^{[0]} x = x.$$

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For finite Ritt order ρ_f the type T_f of $f(s)$ is defined as

$$T_f = \limsup_{\sigma\to\infty} \frac{\log F(\sigma)}{e^{\rho_f \sigma}}.$$

During past decades several authors {see [1],[2],[3],[4]} made close investigations on various properties of the entire Dirichlet series. Therefore with a view to obtain sharper estimation of the growth properties of $f(s)$ when ρ_f is finite,we first introduce the concept of the proximate order and then prove its existence in the line of Shah [6].Let us first define the proximate order of an entire function represented by Dirichlet series.

Definition 1. *Let $f(s)$ be an entire function represented by Dirichlet series* (1) *with finite order ρ_f.A function $\rho(\sigma)$ is said to be a proximate order of f if $\rho(\sigma)$ has the following properties:*

- **:** (i) $\rho(\sigma)$ is non-negative and continuous for $\sigma > \sigma_0$, say,
- **:** (ii) $\rho(\sigma)$ is differentiable for $\sigma \geq \sigma_0$ except possibly at isolated points at which $\rho'(\sigma+0)$ and $\rho'(\sigma-0)$ exist,
- **:** (iii) $\lim\limits_{\sigma\to\infty} \rho(\sigma) = \rho_f$,
- **:** (iv) $\lim\limits_{\sigma\to\infty} \sigma\rho'(\sigma) = 0$ and
- **:** (v) $\limsup\limits_{\sigma\to\infty} \dfrac{\log F(\sigma)}{\exp\{\sigma\rho(\sigma)\}} = 1.$

Since the type T_f is not linked with the proximate order we may expect another comparison function which should closely connect the type and the maximum modulus of an entire function represented by Dirichlet series (1).With this in view we define and prove the existence of such a function in line of Srivastava and Juneja [7] which we call proximate type of $f(s)$.

We now define the proximate type of an entire function represented by Dirichlet series.

Definition 2. *For an entire function $f(s)$ represented by* (1) *with finite order ρ_f and finite type T_f, a function $T(\sigma)$ is said to be a proximate type of f if $T(\sigma)$ has the following properties:*

- **:** (i) $T(\sigma)$ is non-negative and continuous for $\sigma > \sigma_0$, say,
- **:** (ii) $T(\sigma)$is differentiable for $\sigma \geq \sigma_0$ except possibly at isolated points at which $T'(\sigma+0)$ and $T'(\sigma-0)$ exist,
- **:** (iii) $\lim\limits_{\sigma\to\infty} T(\sigma) = T_f$,
- **:** (iv) $\lim\limits_{\sigma\to\infty} \sigma T'(\sigma) = 0$ and
- **:** (v) $\limsup\limits_{\sigma\to\infty} \dfrac{F(\sigma)}{\exp\{T(\sigma)e^{\sigma\rho_f}\}} = 1.$

2. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let $f(s)$ be an entire function represented by Dirichlet series* (1) *with finite Ritt order ρ_f .Then the proximate order $\rho(\sigma)$ of $f(s)$ exists.*

Proof. Let $p(\sigma) = \frac{\log^{[2]} F(\sigma)}{\sigma}$.Then

$$\limsup_{\sigma\to\infty} p(\sigma) = \rho_f .$$

We consider two cases:

Case(I) : Let $p(\sigma) > \rho_f$ for at least a sequences of values of r tending to infinity. we define

$$\phi(\sigma) = \max_{x\geq\sigma}\{p(x)\}.$$

Clearly $\phi(\sigma)$ exists and is non increasing.

Let $R > e^e$ and $p(R) > \rho_f$. Then for $\sigma \geq R_1 > R$ say, we get $p(\sigma) \leq p(R)$ .Since $p(\sigma)$ is continuous, there exists $\sigma_1 \in [R, R_1]$ such that

$$p(\sigma_1) = \max_{R\leq x\leq R_1}\{p(x)\}.$$

Clearly $\sigma_1 > e^e$ and $\phi(\sigma_1) = p(\sigma_1)$ .Such values $\sigma = \sigma_1$ exists for a sequence of values of σ tending to infinity.

Let $\rho(\sigma_1) = \phi(\sigma_1)$ and t_1 be the smallest integer not less than $1 + \sigma_1$ such that $\phi(\sigma_1) > \phi(t_1)$.

We define $\rho(\sigma) = \rho(\sigma_1)$ for $\sigma_1 < \sigma \leq t_1$ .Observing that $\phi(\sigma)$ and $\rho(\sigma_1) - \log^{[2]}\sigma + \log^{[2]} t_1$ are continuous functions of σ, $\rho(\sigma_1) - \log^{[2]}\sigma + \log^{[2]} t_1 > \phi(t_1)$ for $\sigma\,(> t_1)$ sufficiently close to t_1 and $\phi(\sigma)$ is non increasing,we can define u_1 as follows:

$u_1 > t_1$,

$\rho(\sigma) = \rho(\sigma_1) - \log^{[2]}\sigma + \log^{[2]} t_1$ for $t_1 \leq \sigma \leq u_1$,

$\rho(\sigma) = \phi(\sigma)$ for $\sigma = u_1$ and

$\rho(\sigma) > \phi(\sigma)$ for $t_1 \leq \sigma < u_1$.

Let σ_2 be the smallest value of σ for which $\sigma_2 \geq u_1$ and $\phi(\sigma_2) = p(\sigma_2)$ .If $\sigma_2 > u_1$ then let $\rho(\sigma) = \phi(\sigma)$ for $u_1 \leq \sigma \leq \sigma_2$.Since it can be easily shown that $\phi(\sigma)$ is constant in $u_1 \leq \sigma \leq \sigma_2, \rho(\sigma)$ is constant in $u_1 \leq \sigma \leq \sigma_2$.We repeat this process infinitely and obtain that $\rho(\sigma)$ is differentiable in adjacent intervals. Further $\rho'(\sigma) = 0$ or $(-\sigma\log\sigma)^{-1}$ and $\rho(\sigma) \geq \phi(\sigma) \geq p(\sigma)$ for all $\sigma \geq \sigma_1$.Also $\rho(\sigma) = p(\sigma)$ for a sequences of values of σ tending to infinity and $\rho(\sigma)$ is non increasing for $\sigma \geq \sigma_1$ and

$$\rho_f = \limsup_{\sigma\to\infty} p(\sigma) = \lim_{\sigma\to\infty} \phi(\sigma).$$

$$\text{So } \limsup_{\sigma\to\infty} \rho(\sigma) = \liminf_{\sigma\to\infty} \rho(\sigma) = \lim_{\sigma\to\infty} \rho(\sigma) = \rho_f \quad\text{and}\quad \lim_{\sigma\to\infty} \sigma\rho'(\sigma) = 0.$$

Further we have $\log^{[2]} F(\sigma) = \sigma p\left(\sigma\right) = \sigma\rho\left(\sigma\right)$ for a sequence of values of σ tending to infinity and $\log^{[2]} F(\sigma) < \sigma\rho\left(\sigma\right)$ for remaining σ's. Therefore

$$\limsup_{\sigma\to\infty} \frac{\log F(\sigma)}{\exp\left\{\sigma\rho\left(\sigma\right)\right\}} = 1.$$

Continuity of $\rho\left(\sigma\right)$ for $\sigma \geq \sigma_1$ follows from its construction which is complete in case(I).

Case(II) : Let $p\left(\sigma\right) \leq \rho_f$ for all sufficiently large values of r.

In Case(II) we separate two cases:

Sub case (A) : Let $p\left(\sigma\right) = \rho_f$ for at least a sequence of values of σ tending to infinity:

Sub case (B) : Let $p\left(\sigma\right) < \rho_f$ for all sufficiently large values of σ .

In Sub case (A) we take $\rho\left(\sigma\right) = \rho_f$ for all sufficiently large values of σ.

In Sub case (B) let

$$\chi(\sigma) = \max_{X\leq x\leq\sigma}\{p\left(x\right)\},$$

where $X > e^e$ is such that $p\left(\sigma\right) < \rho_f$ whenever $x \geq X$. We note that $\chi(\sigma)$ is non decreasing and for all $\sigma \geq X$ sufficiently large, the roots of $\chi(x) = \rho_f + \log^{[2]} x - \log^{[2]} \sigma$ is less than σ. For a suitable large value $v_1 > X$, we define

$\rho\left(v_1\right) = \rho_f$,

$\rho\left(\sigma\right) = \rho_f + \log^{[2]}\sigma - \log^{[2]} v_1$ for $s_1 \leq \sigma \leq v_1$ where $s_1 < v_1$ is such that $\chi(s_1) = \rho\left(s_1\right)$.In fact s_1 is given by the largest positive root of $\chi(x) = \rho_f + \log^{[2]} x - \log^{[2]} v_1$. If $\chi(s_1) = p\left(s_1\right)$, let $\lambda_1\left(< s_1\right)$ be the upper bound of point λ at which $\chi(\lambda) \neq p\left(\lambda\right)$ and $\lambda < s_1$.Clearly at $\lambda_1, \chi(s_1) = p\left(s_1\right)$ .We define $\rho\left(\sigma\right) = \chi(\sigma)$ for $\lambda_1 \leq \sigma \leq s_1$. It is easy to show that $\chi(\sigma)$ is constant in $\lambda_1 \leq \sigma \leq s_1$ and so $\rho\left(\sigma\right)$ is constant in $\lambda_1 \leq \sigma \leq s_1$. If $\chi(s_1) = p\left(s_1\right)$ we take $\lambda_1 = s_1$.

We choose $v_2 > v_1$ suitably large and let

$\rho\left(v_1\right) = \rho$,

$\rho\left(\sigma\right) = \rho_f + \log^{[2]}\sigma - \log^{[2]} v_2$ for $s_2 \leq \sigma \leq v_2$ where $s_2 < v_2$ is such that $\chi(s_2) = \rho\left(s_2\right)$.If $\chi(s_2) \neq \rho\left(s_2\right)$ let $\rho\left(\sigma\right) = \chi(\sigma)$ for $\lambda_2 \leq \sigma \leq s_2$,where λ_2 has the similar property as that of λ_1.As above $\rho\left(\sigma\right)$ is constant in$[\lambda_2, s_2]$. If $\chi(s_2) = p\left(s_2\right)$ we take $\lambda_2 = s_2$.

Let $\rho\left(\sigma\right) = \rho\left(\lambda_2\right) - \log^{[2]}\sigma + \log^{[2]}\lambda_2$ for $q_1 \leq \sigma \leq \lambda_2$ where $q_1 < \lambda_2$ is the point of intersection of $y = \rho_f$ with $y = \rho\left(\lambda_2\right) - \log^{[2]} x + \log^{[2]}\lambda_2$.It is also possible to choose v_2 so large that $v_1 < q_1$.Let $\rho\left(\sigma\right) = \rho_f$ for $v_1 \leq \sigma \leq q_1$.We repeat this process. Now we can show that for all $\sigma \geq v_1$, $\rho_f \geq \rho\left(\sigma\right) \geq \chi(\sigma) \geq p\left(\sigma\right)$ and $\rho\left(\sigma\right) = p\left(\sigma\right)$ for $\sigma = \lambda_1, \lambda_2, ...$ .So we obtain that

$$\limsup_{\sigma\to\infty} \rho\left(\sigma\right) = \liminf_{\sigma\to\infty} \rho\left(\sigma\right) = \lim_{\sigma\to\infty} \rho\left(\sigma\right) = \rho_f.$$

Since $\log^{[2]} F(\sigma) = \sigma p(\sigma) = \sigma\rho(\sigma)$ for a sequence of values of σ tending to infinity and $\log^{[2]} F(\sigma) < \sigma\rho(\sigma)$ for remaining σ's it follows that

$$\limsup_{\sigma\to\infty} \frac{\log F(\sigma)}{\exp\{\sigma\rho(\sigma)\}} = 1.$$

Also $\rho(\sigma)$ is differentiable in adjacent intervals. Further $\rho'(\sigma) = 0$ or $(\sigma\log\sigma)^{-1}$ and so

$$\lim_{\sigma\to\infty} \sigma\rho'(\sigma) = 0.$$

Continuity of $\rho(\sigma)$ follows from its construction. This completes the proof of the theorem. □

Corollary 1. $\exp\{\sigma\rho(\sigma)\}$ *is an increasing function for* $\sigma > \sigma_0$.

Theorem 2. *Let* $f(s)$ *be an entire function represented by Dirichlet series* (1) *with finite Ritt order* ρ_f *and finite type* T_f *.Then the proximate type* $T(\sigma)$ *of* $f(s)$ *exists.*

Proof. Let

$$s(\sigma) = \frac{\log F(\sigma)}{e^{\rho_f\sigma}}.$$

Then

$$\limsup_{\sigma\to\infty} s(\sigma) = T_f.$$

Then either **Case** (A): Let $s(\sigma) > T_f$ for at least a sequences of values of σ tending to infinity or,

Case (B):Let $s(\sigma) \le T_f$ for all sufficiently large values of σ.

In Case (A) we define

$$\phi(\sigma) = \max_{x\ge\sigma}\{s(x)\}.$$

Clearly $\phi(\sigma)$ exists and is non increasing.

Let $R > e^e$ and $s(R) > T_f$. Then for $\sigma \ge R_1 > R$ say, we get $s(\sigma) \le s(R)$ .Since $s(\sigma)$ is continuous, there exists $\sigma_1 \in [R, R_1]$ such that

$$s(\sigma_1) = \max_{R\le x\le R_1}\{s(x)\}.$$

Clearly $\sigma_1 > e^e$ and $\phi(\sigma_1) = s(\sigma_1)$ .Such values $\sigma = \sigma_1$ exists for a sequence of values of σ tending to infinity.

Let $T(\sigma_1) = \phi(\sigma_1)$ and p_1 be the smallest integer not less than $1+\sigma_1$ such that $\phi(\sigma_1) > \phi(p_1)$.

We define $T(\sigma) = T(\sigma_1)$ for $\sigma_1 < \sigma \le p_1$ .Observing that $\phi(\sigma)$ and $T(\sigma_1) - \log^{[2]}\sigma + \log^{[2]} p_1$ are continuous functions of σ, $T(\sigma_1) - \log^{[2]}\sigma + \log^{[2]} p_1 > \phi(p_1)$ for $\sigma (> p_1)$ sufficiently close to p_1 and $\phi(\sigma)$ is non increasing,we can define u_1 as follows:

$u_1 > p_1$,

$T(\sigma) = T(\sigma_1) - \log^{[2]}\sigma + \log^{[2]} p_1$ for $p_1 \le \sigma \le u_1$,

$T(\sigma) = \phi(\sigma)$ for $\sigma = u_1$ and

$T(\sigma) > \phi(\sigma)$ for $p_1 \leq \sigma < u_1$.

Let σ_2 be the smallest value of σ for which $\sigma_2 \geq u_1$ and $\phi(\sigma_2) = s(\sigma_2)$.If $\sigma_2 > u_1$ then let $T(\sigma) = \phi(\sigma)$ for $u_1 \leq \sigma \leq \sigma_2$.Since it can be easily shown that $\phi(\sigma)$ is constant in $u_1 \leq \sigma \leq \sigma_2, T(\sigma)$ is constant in $u_1 \leq \sigma \leq \sigma_2$.We repeat this process infinitely and obtain that $T(\sigma)$ is differentiable in adjacent intervals. Further $T'(\sigma) = 0$ or $(-\sigma \log \sigma)^{-1}$ and $T(\sigma) \geq \phi(\sigma) \geq s(\sigma)$ for all $\sigma \geq \sigma_1$.Also $T(\sigma) = s(\sigma)$ for a sequences of values of σ tending to infinity and $T(\sigma)$ is non increasing for $\sigma \geq \sigma_1$ and

$$T_f = \limsup_{\sigma \to \infty} s(\sigma) = \lim_{\sigma \to \infty} \phi(\sigma).$$

$$\text{So } \limsup_{\sigma \to \infty} T(\sigma) = \liminf_{\sigma \to \infty} T(\sigma) = \lim_{\sigma \to \infty} T(\sigma) = T_f \quad \text{and} \quad \lim_{\sigma \to \infty} \sigma T'(\sigma) = 0.$$

Further we have $F(\sigma) = \exp\{s(\sigma) e^{\rho_f \sigma}\} = \exp\{T(\sigma) e^{\rho_f \sigma}\}$ for a sequence of values of σ tending to infinity and $F(\sigma) < \exp\{T(\sigma) e^{\rho_f \sigma}\}$ for remaining σ's,.Therefore

$$\limsup_{\sigma \to \infty} \frac{F(\sigma)}{\exp\{T(\sigma) e^{\sigma \rho_f}\}} = 1.$$

Continuity of $T(\sigma)$ for $\sigma \geq \sigma_1$ follows from its construction which is complete in Case(A).

In Case(B) we separate two cases:

Sub case (I):Let $s(\sigma) = \sigma_f$ for at least a sequence of values of σ tending to infinity:

Sub case (II):Let $s(\sigma) < \sigma_f$ for all sufficiently large values of σ .

In Sub case (I) we take $T(\sigma) = \sigma_f$ for all sufficiently large values of σ.

In Sub case (II) let

$$\xi(\sigma) = \max_{X \leq x \leq \sigma} \{s(x)\},$$

where $X > e^e$ is such that $s(\sigma) < \sigma_f$ whenever $x \geq X$. We note that $\xi(\sigma)$ is non decreasing and for all $\sigma \geq X$ sufficiently large, the roots of $\xi(x) = \rho_f + \log^{[2]} x - \log^{[2]} \sigma$ is less than σ. For a suitable large value $v_1 > X$, we define

$T(v_1) = \sigma_f$,

$T(\sigma) = \sigma_f + \log^{[2]} \sigma - \log^{[2]} v_1$ for $s_1 \leq \sigma \leq v_1$ where $s_1 < v_1$ is such that $\xi(s_1) = T(s_1)$.In fact s_1 is given by the largest positive root of $\xi(x) = \rho_f + \log^{[2]} x - \log^{[2]} v_1$. If $\xi(s_1) = T(s_1)$ let $\omega_1 (< s_1)$ be the upper bound of point ω at which $\xi(\omega) \neq s(\omega)$ and $\omega < s_1$.Clearly at $\omega_1, \xi(s_1) = s(s_1)$.We define $T(\sigma) = \xi(\sigma)$ for $\omega_1 \leq \sigma \leq s_1$. It is easy to show that $\xi(\sigma)$ is constant in $\omega_1 \leq \sigma \leq s_1$ and so $T(\sigma)$ is constant in $\omega_1 \leq \sigma \leq s_1$. If $\xi(s_1) = s(s_1)$ we take $\omega_1 = s_1$.

We choose $v_2 > v_1$ suitably large and let

$T(v_1) = \sigma_f$,

$T(\sigma) = \sigma_f + \log^{[2]} \sigma - \log^{[2]} v_2$ for $s_2 \leq \sigma \leq v_2$ where $s_2 < v_2$ is such that $\xi(s_2) = T(s_2)$.If $\xi(s_2) \neq T(s_2)$ let $T(\sigma) = \xi(\sigma)$ for $\omega_2 \leq \sigma \leq s_2$,where ω_2 has the similar property as that of ω_1.As above $T(\sigma)$ is constant in$[\omega_2, s_2]$. If $\xi(s_2) = T(s_2)$ we take $\omega_2 = s_2$.

Let $T(\sigma) = T(\omega_2) - \log^{[2]}\sigma + \log^{[2]}\omega_2$ for $q_1 \leq \sigma \leq \omega_2$ where $q_1 < \omega_2$ is the point of intersection of $y = T$ with $y = T(\omega_2) - \log^{[2]} x + \log^{[2]}\omega_2$.It is also possible to choose v_2 so large that $v_1 < q_1$.Let $T(\sigma) = \sigma_f$ for $v_1 \leq \sigma \leq q_1$.We repeat this process. Now we can show that for all $\sigma \geq v_1$, $\sigma_f \geq T(\sigma) \geq \xi(\sigma) \geq s(\sigma)$ and $T(\sigma) = s(\sigma)$ for $\sigma = \omega_1, \omega_2, \ldots$ .So we get that

$$\limsup_{\sigma\to\infty} T(\sigma) = \liminf_{\sigma\to\infty} T(\sigma) = \lim_{\sigma\to\infty} T(\sigma) = \sigma_f.$$

Since$F(\sigma) = \exp\{s(\sigma)e^{\rho_f\sigma}\} = \exp\{T(\sigma)e^{\rho_f\sigma}\}$ for a sequence of values of σ tending to infinity and $F(\sigma) < \exp\{T(\sigma)e^{\rho_f\sigma}\}$ for remaining σ's,we get that

$$\limsup_{\sigma\to\infty} \frac{F(\sigma)}{\exp\{T(\sigma)e^{\sigma\rho_f}\}} = 1.$$

Also $T(\sigma)$ is differentiable in adjacent intervals. Further $T'(\sigma) = 0$ or $(\sigma\log\sigma)^{-1}$ and so

$$\lim_{\sigma\to\infty} \sigma T'(\sigma) = 0.$$

Continuity of $T(\sigma)$ follows from its construction. This completes the proof of the theorem. □

Theorem 3. *Let $T(\sigma)$ be the proximate type of $f(s)$.Then*

$$\liminf_{\sigma\to\infty} \frac{\log T(\sigma)}{\sigma} = 0.$$

Proof. As $\limsup_{\sigma\to\infty} \frac{F(\sigma)}{\exp\{T(\sigma)e^{\sigma\rho_f}\}} = 1$, for arbitrary $\varepsilon > 0$ and for sequence of values of σ we get

$$\begin{aligned}
(1-\varepsilon)\exp\{T(\sigma)e^{\sigma\rho_f}\} &\leq F(\sigma)\\
i.e., \log T(\sigma) + \sigma\rho_f + O(1) &\leq \log^{[2]} F(\sigma)\\
i.e., \frac{\log T(\sigma)}{\sigma} + \rho_f + \frac{O(1)}{\sigma} &\leq \frac{\log^{[2]} F(\sigma)}{\sigma}\\
i.e., \liminf_{\sigma\to\infty} \frac{\log T(\sigma)}{\sigma} + \rho_f &\leq \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma} = \rho_f
\end{aligned}$$

$$i.e., \liminf_{\sigma\to\infty} \frac{\log T(\sigma)}{\sigma} \leq 0.$$

Since $T(\sigma)$ is non negative it follows that

$$\liminf_{\sigma\to\infty} \frac{\log T(\sigma)}{\sigma} = 0.$$

This completes the proof of the theorem. □

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