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# MERCERIAN THEOREM FOR FOUR DIMENSIONAL MATRICES

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Abstract. Let $A = (a_{nk})$ be an infinite matrix and let $c$ and $c_A$ be the space of all convergent sequences with complex terms and convergence domain of $A$, respectively. In 1907, Mercer proved in [*On the limits of real variants*, Proc. London Math. Soc. **2 (1)** (1907), no. 5, 206–224.] that $c = c_A$ which is called a *Mercerian theorem*. In this paper, we give the corresponding theorem for a four dimensional matrix and the space of convergent double sequences in the Pringsheim's sense.

## 1. Introduction

We denote the set of all complex valued double sequences by $\Omega$ which is a linear space with coordinatewise addition and scalar multiplication. Any linear subspace of $\Omega$ is called as *a double sequence space*. A double sequence $x = (x_{mn})$ of complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$ which is a Banach space with the norm $\|\cdot\|_\infty$. Consider the sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$, then we say that the double sequence $x$ is *convergent* in the *Pringsheim's sense* to the limit $l$ and write $p - \lim x_{mn} = l$, [1]; where $\mathbb{C}$ denotes the complex field. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. Indeed following Boos [2, p. 16], if we define the sequence $x = (x_{mn})$ by

$$x_{mn} := \begin{cases} n & , \quad m = 0,\ n \in \mathbb{N}, \\ 0 & , \quad m \geq 1,\ n \in \mathbb{N}, \end{cases}$$

then it is trivial that $x \in \mathcal{C}_p - \mathcal{M}_u$, since $p - \lim x_{mn} = 0$ but $\|x\|_\infty = \infty$. So, we can consider the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the

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Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is convergent in the ordinary sense with respect to each index and denote the space of all such sequences by $\mathcal{C}_r$. Also by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$, we denote the spaces of all double null sequences contained in the sequence spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively. Móricz [3] proved that $\mathcal{C}_{bp}$, $\mathcal{C}_{bp0}$, $\mathcal{C}_r$ and $\mathcal{C}_{r0}$ are Banach spaces with the norm $\|\cdot\|_\infty$. The reader can refer to [4, 5, 6, 7, 8, 9, 10] for further details about the double sequences, four dimensional matrices and related topics.

Boos, Leiger and Zeller [11] introduced and investigated the notion of $e-$convergence of double sequences, which is essentially weaker than the convergence in the Pringsheim's sense. A double sequence $x = (x_{mn})$ is said to be $e-$convergent to a number $l$ if

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 \; \exists m_n \in \mathbb{N} \text{ such that } m \geq m_n \Rightarrow \; |x_{mn} - l| \leq \varepsilon.$$

If $x$ is $e-$convergent and, in addition, $(x_{mn})_{m\in\mathbb{N}}$ is bounded for every $n \in \mathbb{N}$, or equivalently the limit $\lim_{m\to\infty} x_{mn}$ exists for every fixed $n \in \mathbb{N}$, then $x$ is said to be $be-$convergent and $c-$convergent, respectively. Evidently, the $be-$ and $c-$convergence generalize the $bp-$ and $r-$convergence, respectively. Note that in the case of the $c-$convergence also the limit $\lim_{n\to\infty}\lim_{m\to\infty} x_{mn}$ exists and is equal to the $e-$limit.

The $\beta(\vartheta)$-*dual* $\lambda^{\beta(\vartheta)}$ with respect to the $\vartheta$-convergence of a double sequence space $\lambda$ is defined by

$$\lambda^{\beta(\vartheta)} := \left\{ (a_{kl}) \in \Omega : \vartheta - \sum_{k,l=0}^{\infty} a_{kl}x_{kl} \; \text{ exists for all } (x_{kl}) \in \lambda \right\}.$$

Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, a double sequence $x = (x_{kl})$ is said to be in the application domain of $A$ with respect to $\vartheta$ if and only if

$$(Ax)_{mn} = \vartheta - \sum_{k,l=0}^{\infty} a_{mnkl}x_{kl} \tag{1}$$

exists for each $m, n \in \mathbb{N}$. We define the $\vartheta$-*summability domain* $\lambda_A^{(\vartheta)}$ of $A$ in a double sequence space $\lambda$ by

$$\lambda_A^{(\vartheta)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left( \vartheta - \sum_{k,l=0}^{\infty} a_{mnkl}x_{kl} \right)_{m,n\in\mathbb{N}} \text{ exists and is in } \lambda \right\}.$$

Let $\lambda$ and $\mu$ be two spaces of double sequences, and $A$ be a four dimensional matrix. Then, we say with the notation (1) that $A$ maps the space $\lambda$ into the space $\mu$ if $\lambda \subset \mu_A^{(\vartheta)}$ and we denote the set of all four dimensional matrices, transforming the space $\lambda$ into the space $\mu$, by $(\lambda : \mu)$. Thus, $A = (a_{mnkl}) \in (\lambda : \mu)$ if and only if the double series on the right side of (1) converges in the sense of $\vartheta$ for each $m, n \in \mathbb{N}$, i.e, $A_{mn} \in \lambda^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$; where $A_{mn} = (a_{mnkl})_{k,l \in \mathbb{N}}$ for all $m, n \in \mathbb{N}$. In the special case $\lambda = \mathcal{C}_\vartheta$ the set

$$(\mathcal{C}_\vartheta)_A = \{x = (x_{kl}) \in \Omega : Ax \in \mathcal{C}_\vartheta\}$$

is called the $\vartheta-$convergence domain of $A$. Here and after, unless stated otherwise we assume that $\vartheta$ denotes any of the symbols $p$, $bp$, $r$, $e$, $be$ or $c$. We say that $A$ is *$C_\vartheta$-conservative* if $\mathcal{C}_\vartheta \subset (\mathcal{C}_\vartheta)_A$, and is *$C_\vartheta$-regular* if it is $C_\vartheta$-conservative and $\vartheta - \lim_A x = \vartheta - \lim_{m,n\to\infty} (Ax)_{mn} = \vartheta - \lim_{m,n\to\infty} x_{mn}$, where $x = (x_{mn}) \in \mathcal{C}_\vartheta$.

For all $m, n, k, l \in \mathbb{N}$, we say that $A = (a_{mnkl})$ is *a triangular matrix* if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both, [12]. By following Adams [12], we can say that a triangular matrix $A = (a_{mnkl})$ is called a *triangle* if $a_{mnmn} \neq 0$ for all $m, n \in \mathbb{N}$. Referring to Cooke [13, Remark (a), p. 22], one can conclude that every triangle matrix has an unique inverse which is also a triangle.

Following Zeltser [14], we define the double sequence $\mathbf{e}^{\mathbf{kl}} = \left(e_{mn}^{kl}\right)$ by

$$e_{mn}^{kl} := \begin{cases} 1 & , \quad (k,l) = (m,n), \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, l, m, n \in \mathbb{N}$.

We use the notation $\sim$ as in [19], that is, "$f \sim g$" means "$f/g \to 1$".

**Definition 1.1.** [15, Definition 1.7.4, p. 12] If $c_B \supseteq c_A$, then $B$ is said to be stronger than $A$.

**Definition 1.2.** [15, Definition 1.7.13, p. 14] A matrix $A$ is said to be Mercerian if $c_A = c$.

**Definition 1.3.** [16, 17] Any four dimensional matrix is said to be $RH-$regular if it maps every bounded $p-$convergent sequence into a $p-$convergent sequence with the same $p-$limit.

**Theorem 1.1.** [16, 17] A four dimensional matrix $A = (a_{mnkl})$ is $RH-regular$ if and only if

$RH_1$ : $p-\lim_{m,n\to\infty} a_{mnkl} = 0$ for each $k,l \in \mathbb{N}$,

$RH_2$ : $p-\lim_{m,n\to\infty} \sum_{k,l=0}^{\infty} a_{mnkl} = 1$,

$RH_3$ : $p-\lim_{m,n\to\infty} \sum_{k=0}^{\infty} |a_{mnkl}| = 0$ for each $l \in \mathbb{N}$,

$RH_4$ : $p-\lim_{m,n\to\infty} \sum_{l=0}^{\infty} |a_{mnkl}| = 0$ for each $k \in \mathbb{N}$,

$RH_5$ : $\sum_{k,l=0}^{\infty} |a_{mnkl}|$ is p-convergent,

$RH_6$ : there exist finite positive integers $M$ and $N$ such that $\sum_{k,l>N} |a_{mnkl}| < M$.

Now, we give our definitions for four dimensional matrices.

**Definition 1.4.** Let $A = (a_{mnkl})$ and $B = (b_{mnkl})$ be two four dimensional matrices. If every $A$ summable sequence is also $B$ summable, then $B$ is said to be stronger than $A$ and we write $(\mathcal{C}_\vartheta)_B \supseteq (\mathcal{C}_\vartheta)_A$.

**Definition 1.5.** A four dimensional matrix $A = (a_{mnkl})$ is said to be Mercerian if $(\mathcal{C}_\vartheta)_A = \mathcal{C}_\vartheta$.

Let $A = (a_{nk})$ be an infinite matrix and let $c$ and $c_A$ be the space of all convergent sequences with complex terms and convergence domain of $A$, respectively. The result given by Mercer for the space $c$ which proves that $c = c_A$, is called a *Mercerian theorem*, [18]. Hardy [19], Maddox [20] described Mercer's result as follows. Let $x = (x_k)$ be an ordinary sequence and consider the transformation $A$ defined by

$$(Ax)_n = \alpha x_n + \frac{1-\alpha}{n+1} \sum_{k=0}^{n} x_k$$

for all $n \in \mathbb{N}$, where $\alpha > 0$ is a real number. Then, $(Ax)_n \to l$ implies $x_k \to l$.

It is well-known that if the four dimensional matrix $A = (a_{mnkl})$ is in the class $(C_\vartheta : C_\vartheta)$, then the inclusion $C_\vartheta \subset (C_\vartheta)_A$ holds. Note that the question "When does $C_\vartheta = (C_\vartheta)_A$ hold?" is still open problem. In this paper, we essentially study to solve this problem with referring Hardy [19].

## 2. Main Results

In this section, we give the Mercerian theorem for four dimensinal matrices and the space of convergent double sequences in the Pringsheim's sense together with the results on the associativity of the products $t(Ax)$ and $B(Ax)$, where $t, x \in \Omega$ and $A, B$ are the four dimensional matrices.

**Theorem 2.1.** *Let $x = (x_{mn})$ be a double sequence and consider the double sequence $s = (s_{mn})$ defined by $s_{mn} = \sum_{k,l=0}^{m,n} x_{kl}/[(m+1)(n+1)]$ for all $m, n \in \mathbb{N}$. If $\alpha > 0$, $\{\alpha x_{mn} + (1-\alpha)s_{mn}\}$ bounded and*

$$\vartheta - \lim_{m,n\to\infty} [\alpha x_{mn} + (1-\alpha)s_{mn}] = l, \tag{2}$$

*then $\vartheta - \lim_{m,n\to\infty} x_{mn} = l$.*

*Proof.* Let the double sequence $z = (z_{mn})$ be defined by

$$z_{mn} = \alpha x_{mn} + (1-\alpha)s_{mn}$$

for all $m, n \in \mathbb{N}$. We assume that $s_{-1,n} = s_{m,-1} = s_{-1,-1} = 0$ for all $m, n \in \mathbb{N}$. Since

$$s_{mn} = \frac{1}{(m+1)(n+1)} \sum_{k,l=0}^{m,n} x_{kl}$$

for all $m, n \in \mathbb{N}$, we have

$$x_{mn} = (m+1)(n+1)s_{m,n} - m(n+1)s_{m-1,n} - (m+1)ns_{m,n-1} + mns_{m-1,n-1}$$

for all $m, n \in \mathbb{N}$. Thus, we can write

$$\begin{aligned} z_{mn} &= [(mn+m+n)\alpha + 1]s_{mn} - m(n+1)\alpha s_{m-1,n} \\ &- (m+1)n\alpha s_{m,n-1} + mn\alpha s_{m-1,n-1}. \end{aligned} \tag{3}$$

We choose the sequences $q = (q_k)$ and $t = (t_l)$ of non-negative numbers which are not all zero with $q_0 = t_0 = 1$ and $q_1 \neq 1$ so as to satisfy

$$\begin{aligned} &[(kl+k+l)\alpha + 1]q_k t_l - (k+1)(l+1)\alpha q_k t_{l+1} - (k+1)(l+1)\alpha q_{k+1} t_l + \\ &(k+1)(l+1)\alpha q_{k+1} t_{l+1} = 0 \end{aligned} \tag{4}$$

for $0 \leq k \leq m-1$ and $0 \leq l \leq n-1$,

$$[(kn+k+n)\alpha + 1]q_k t_n - (k+1)(n+1)\alpha q_{k+1} t_n = 0 \tag{5}$$

for $0 \leq k \leq m-1$, and

$$[(ml+m+l)\alpha + 1]q_m t_l - (m+1)(l+1)\alpha q_m t_{l+1} = 0 \tag{6}$$

for $0 \leq l \leq n-1$, for all $m, n, k, l \in \mathbb{N}$. Therefore, we can write from the relations (4)-(6) that

$$\begin{aligned} t_n &= \frac{n\alpha q_1 - [(n-1)\alpha + 1]}{n\alpha(q_1 - 1)} \cdots \left[\frac{3\alpha q_1 - (2\alpha+1)}{3\alpha(q_1-1)}\right] \left[\frac{2\alpha q_1 - (\alpha+1)}{2\alpha(q_1-1)}\right] \left[\frac{\alpha q_1 - 1}{\alpha(q_1-1)}\right], \\ q_m &= \frac{1}{m!} q_1(q_1+1)(q_1+2)\cdots(q_1+m-1). \end{aligned}$$

We easily see that $q_m > 0$ for all $m \in \mathbb{N}$ and it always exists. But if we take either $q_1 < 1/\alpha < 1$ or $1 < 1/\alpha < q_1$, we can say that $t_n > 0$ for all $n \in \mathbb{N}$ and it always exists, as well.

Then, we derive with a straightforward calculation that

$$\sum_{k,l=0}^{m,n} q_k t_l \sim [(mn+m+n)\alpha+1] q_m t_n \tag{7}$$

for all $m, n \in \mathbb{N}$ by the relations (4)-(6).

Multiplying the equality (3) by $q_0t_0, q_0t_1, \ldots, q_1t_0, q_1t_1, \ldots, q_mt_0, q_mt_1, \ldots$, adding, and using the relation (7) and considering the $RH-$regularity of the Riesz mean $R^{qt}$ (see [21, Theorem 2.8]), we obtain that

$$\vartheta - \lim_{m,n\to\infty} s_{mn} = \vartheta - \lim_{m,n\to\infty} \sum_{k,l=0}^{m,n} \frac{q_k t_l}{[(mn+m+n)\alpha+1] q_m t_n} z_{kl} = l \tag{8}$$

and it follows from (2) and (8) that $\vartheta - \lim_{m,n\to\infty} x_{mn} = l$. This completes the proof. □

**Theorem 2.2.** *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, $A \in (\mathcal{M}_u : \mathcal{M}_u)$ if and only if*

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l=0}^{\infty} |a_{mnkl}| < \infty. \tag{9}$$

*Proof.* Let $A \in (\mathcal{M}_u : \mathcal{M}_u)$. Then, $Ax$ exists and belongs to $\mathcal{M}_u$ for all $x \in \mathcal{M}_u$, and $A_{mn} \in \mathcal{M}_u^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$. Hence, $\vartheta - \lim_{m,n\to\infty}(Ax)_{mn}$ exists and $\sup_{m,n\in\mathbb{N}} |(Ax)_{mn}| < \infty$ for all $x \in \mathcal{M}_u$. Putting $Ax = \{(Ax)_{mn}\}$ and using the Banach-Steinhaus theorem, we see that the condition (9) is necessary.

Conversely, suppose that the condition (9) holds and take any $x = (x_{kl}) \in \mathcal{M}_u$. Then, $A_{mn} \in \mathcal{M}_u^{\beta(\vartheta)}$ for each $m, n \in \mathbb{N}$ which implies the existence of $Ax$. Let $m, n \in \mathbb{N}$ be fixed. Then, since

$$\begin{aligned}
\left| \sum_{k,l=0}^{\infty} a_{mnkl} x_{kl} \right| &\leq \sum_{k,l=0}^{\infty} |a_{mnkl} x_{kl}| \\
&= \sum_{k,l=0}^{\infty} |a_{mnkl}||x_{kl}| \\
&\leq \|x\|_{\infty} \sum_{k,l=0}^{\infty} |a_{mnkl}|
\end{aligned}$$

one can obtain by taking supremum over $m, n \in \mathbb{N}$ that

$$\|Ax\|_\infty \leq \|x\|_\infty \sup_{m,n\in\mathbb{N}} \sum_{k,l=0}^{\infty} |a_{mnkl}| < \infty,$$

which leads us to the fact that $Ax \in \mathcal{M}_u$, as desired.

This step completes the proof. $\square$

The expressions $t(Ax)$ and $(tA)x$ arise often in summability, where $x, t \in \Omega$ and $A = (a_{mnkl})$ is a four dimensional matrix. We define the double sequence $b = (b_{kl})$ by

$$b_{kl} = tA^{kl} = \sum_{m,n=0}^{\infty} t_{mn} a_{mnkl} \tag{10}$$

for $k, l \in \mathbb{N}$, where $A^{kl} = (a_{mnkl})_{m,n=0}^{\infty}$. Then,

$$t(Ax) = \sum_{m,n=0}^{\infty} \sum_{k,l=0}^{\infty} t_{mn} a_{mnkl} x_{kl}$$

and

$$bx = \sum_{k,l=0}^{\infty} \sum_{m,n=0}^{\infty} t_{mn} a_{mnkl} x_{kl}$$

may be different even if $t = (t_{kl}) \in \mathcal{L}_u$, $A$ is a $RH-$regular triangle, $x \in (\mathcal{C}_\vartheta)_A$ and both numbers exists, where $\mathcal{L}_u$ is the space of absolutely convergent double series. Let us define $t = (t_{kl})$ such that $t_{kl}$ is $\frac{1}{(k+1)(k+2)(l+1)(l+2)}$ in the first column to $(l-1)th$ column and is zero otherwise, that is,

$$(t_{kl}) = \begin{array}{c} \begin{array}{cccccccc} \phantom{\frac{1}{(k+1)(k+2)\cdot 2}} & \phantom{\cdots} & \phantom{\frac{1}{(k+1)(k+2)(l-1)l}} & (l-1)th\ \text{column} & & & & \end{array} \\ \begin{pmatrix} \frac{1}{2\cdot 2} & \cdots & \frac{1}{2\cdot(l-1)\cdot l} & \frac{1}{2\cdot l\cdot(l+1)} & 0 & 0 & 0 & \cdots \\ \frac{1}{2\cdot 3\cdot 2} & \cdots & \frac{1}{2\cdot 3\cdot(l-1)\cdot l} & \frac{1}{2\cdot 3\cdot l\cdot(l+1)} & 0 & 0 & 0 & \cdots \\ \frac{1}{3\cdot 4\cdot 2} & \cdots & \frac{1}{3\cdot 4\cdot(l-1)\cdot l} & \frac{1}{3\cdot 4\cdot l\cdot(l+1)} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \frac{1}{(k+1)(k+2)\cdot 2} & \cdots & \frac{1}{(k+1)(k+2)(l-1)l} & \frac{1}{(k+1)(k+2)l(l+1)} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \end{array}$$

and let $A = C_1$, where $C_1 = (c_{mnkl})$ denotes the four dimensional Cesàro matrix of order one. The matrix $C_1$ is a $RH-$regular triangle matrix. Then, we have $b = 0$. If we choose $x = (x_{kl})$ such that $C_1 x = \mathbf{e^{kl}}$, then we obtain

$$t(C_1 x) = \sum_{m,n=0}^{\infty} \sum_{k,l=0}^{\infty} t_{mn} c_{mnkl} x_{kl} = \sum_{m,n=0}^{\infty} t_{mn} = 1 - \frac{1}{l+1} \neq 0$$

for all natural numbers $l$.

**Theorem 2.3.** **(Associativity of $\mathbf{t}(\mathbf{Ax})$)** Let $x, t \in \Omega$, $A = (a_{mnkl})$ be an infinite matrix and $b$ be the double sequence given by the relation (10). Then we have $t(Ax) = bx$, if one of the following statements holds:

(i) $t \in \varphi$ and $x \in \Omega_A^{(\vartheta)}$, where $\varphi$ denotes the space of all finitely non-zero double sequences.
(ii) $t \in \mathcal{L}_u$, $A \in (\mathcal{M}_u : \mathcal{M}_u)$ and $x \in \mathcal{M}_u$.

*Proof.* Since the proof is easily obtained in the similar way used in Wilansky [15, Theorem 1.4.4, p. 8], we omit the detail. □

**Theorem 2.4.** **(Associativity of $\mathbf{B}(\mathbf{Ax})$)** Let $x = (x_{kl}) \in \Omega$, $A = (a_{mnkl})$ and $B = (b_{mnkl})$ be four dimensional infinite matrices. Then $B(Ax)$ and $(BA)x$ exist, and $B(Ax) = (BA)x$, if one of the following statements holds:

(i) $B_{mn} \in \varphi$ for each $m, n \in \mathbb{N}$ and $x \in \Omega_A^{(\vartheta)}$.
(ii) $B_{mn} \in \mathcal{L}_u$ for each $m, n \in \mathbb{N}$, $A \in (\mathcal{M}_u : \mathcal{M}_u)$ and $x \in \mathcal{M}_u$.

*Proof.* This is an immediate consequence of Theorem 2.3 by taking $B_{mn}$ instead of $t$. □

**Theorem 2.5.** *Let $A$ and $B$ be four dimensional triangles. Then, $B$ is stronger than $A$ if and only if $B^{-1}A$ is $C_\vartheta$-conservative.*

*Proof.* Let $A$ and $B$ be four dimensional triangles. Then, $A^{-1}$ and $B^{-1}$ exist.

We assume that $B$ is stronger than $A$. Let $x \in C_\vartheta$ be given. We take $y = A^{-1}x$. Since $x \in C_\vartheta$ and $Ay = A(A^{-1}x) = (AA^{-1})x = x$ by Part (i) of Theorem 2.4, $y \in (C_\vartheta)_A$. Then, we get that $y \in (C_\vartheta)_B$. Hence, $By \in C_\vartheta$. Also $By = B(A^{-1}x) = (BA^{-1})x$, that is, $x \in (C_\vartheta)_{BA^{-1}}$. Therefore, $C_\vartheta \subset (C_\vartheta)_{BA^{-1}}$, as desired.

Conversely, we assume that $B^{-1}A$ is $C_\vartheta$-*conservative*. Let $x \in (C_\vartheta)_A$ be given. Then, we have $Ax \in C_\vartheta$. Applying by Part (i) of Theorem 2.4, again, and using the assumption that $B^{-1}A$ is $C_\vartheta$-*conservative*, we conclude $Bx = B(A^{-1}A)x = (BA^{-1})(Ax) \in C_\vartheta$. Hence, $x \in (C_\vartheta)_B$. So, we can say that $B$ is stronger than $A$ which completes the proof. □

Now, we can give the following result which is the immediate consequence of Theorem 2.5 with $B = I$:

**Corollary 2.6.** *A $C_\vartheta$-conservative four dimensional triangle matrix $A$ is Mercerian if and only if $A^{-1}$ is $C_\vartheta$-conservative.*

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