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SOME GEOMETRIC PROPERTIES OF THE DOMAIN OF THE TRIANGLE $\widetilde{A}$ IN THE SEQUENCE SPACE $\ell(p)$*

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Abstract. The sequence space $\ell(\widetilde{A}, p)$ of non-absolute type is the domain of the triangle matrix $\widetilde{A}$ defined by the strictly increasing sequence $\lambda = (\lambda_n)$ of positive real numbers tending to infinity in the sequence space $\ell(p)$, where $\ell(p)$ denotes the space of all sequences $x = (x_k)$ such that $\sum_k |x_k|^{p_k} < \infty$ and were defined by Maddox in [*Spaces of strongly summable sequences*, Quart. J. Math. Oxford (2) **18** (1967), 345–355]. The main purpose of this paper is to investigate the geometric properties of the space $\ell\big(\widetilde{A}, p\big)$, like rotundity, Kadec-Klee property.

1. Introduction

By ω, we denote the space of all sequences with complex elements which contains ϕ, the set of all finitely non-zero sequences, that is,

$$\omega := \{x = (x_k) : x_k \in \mathbb{C} \ \text{ for all } \ k \in \mathbb{N}\},$$

where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. By a sequence space, we understand a linear subspace of the space ω. We write ℓ_∞, c, c_0 and ℓ_p for the classical sequence spaces of all bounded, convergent, null and absolutely p-summable sequences which are the Banach spaces with the norms $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and $\|x\|_p = (\sum_k |x_k|^p)^{1/p}$; respectively, where $1 \le p < \infty$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. Also by bs and cs, we denote the spaces of all bounded and convergent series, respectively. bv is the space consisting of all sequences (x_k) such that $(x_k - x_{k+1})$ in ℓ_1 and bv_0 is the intersection of the spaces bv and c_0.

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A linear topological space X over the real field $\mathbb{R}$ is said to be a *paranormed space* if there is a subadditive function $g : X \to \mathbb{R}$ satisfying the following conditions for all $x, y \in X$:

(i) $g(\theta) = 0$.
(ii) $g(x) = g(-x)$.
(iii) Scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $\ell(p)$ was defined by Maddox [2] (see also Simons [3] and Nakano [4]) as follows:

$$\ell(p) := \left\{ x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty \right\},\ (0 < p_k \leq H < \infty)$$

which is complete paranormed space paranormed by

$$g(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M}.$$

We assume throughout that $p_k^{-1} + (p'_k)^{-1} = 1$ provided $\inf p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

The beta-dual λ^β of a sequence space λ is defined by

$$\lambda^\beta \quad = \quad \{x = (x_k) \in \omega : xy = (x_k y_k) \in cs \ \text{ for all } \ y = (y_k) \in \lambda\}.$$

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1.1}$$

provided the series on the right side of (1.1) converges for each $n \in \mathbb{N}$. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if Ax exists, i.e. $A_n \in \lambda^\beta$ for all $n \in \mathbb{N}$ and is in μ for all $x \in \lambda$, where A_n denotes the sequence in the n-th row of A.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for triangles A, B and any sequence x. Further, a triangle matrix U uniquely has an inverse $U^{-1} = V$ which is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A := \big\{x = (x_k) \in \omega : Ax \in \lambda\big\}.$$

If A is triangle, then one can easily observe that the sequence spaces λ_A and λ are linearly isomorphic, i.e. $\lambda_A \cong \lambda$.

We consider the strictly increasing sequence $\lambda = (\lambda_k)_{k=0}^{\infty}$ of positive reals tending to ∞, that is

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Via the sequence $\lambda = (\lambda_k)_{k\in\mathbb{N}}$, we define the triangle matrix $\widetilde{A} = (\widetilde{a}_{nk})$ by

$$\widetilde{a}_{nk}(\lambda) = \begin{cases} \frac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_n - \lambda_{n-1}} & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. It is easy to show that $\widetilde{A}$ is a regular matrix and a straightforward calculation yields that the inverse $\widetilde{A}^{-1} = \{b_{nk}(\lambda)\}$ of the matrix $\widetilde{A}$ is given by the following double band matrix as

$$b_{nk}(\lambda) = \begin{cases} (-1)^{n-k} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} & , \quad n-1 \le k \le n, \\ 0 & , \quad 0 \le k < n-1 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. We study some geometric properties of the sequence space $\ell(\widetilde{A}, p)$ of non-absolute type which is the domain of the triangle matrix $\widetilde{A}$ in the sequence space $\ell(p)$, that is

$$\ell(\widetilde{A}, p) := \left\{ (x_k) \in \omega : \sum_k \left| \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^{p_k} < \infty \right\}$$

which is a complete linear metric space paranormed by the paranorm

$$g_1(x) = \left(\sum_k \left| \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^{p_k} \right)^{1/M}$$

and has the AK property. In the special case $p_k = p$ for all $k \in \mathbb{N}$, the space $\ell(\widetilde{A}, p)$ is reduced to the space $\ell_p(\widetilde{A})$, i.e.,

$$\ell_p(\widetilde{A}) := \left\{ (x_k) \in \omega : \sum_k \left| \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^{p} < \infty \right\}, \ (0 < p < \infty)$$

which is a BK-space with the norm

$$\|x\| = \left(\sum_k \left| \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^{p} \right)^{1/p}, \text{ where } \ 1 \le p < \infty$$

and is a complete p-normed space with the p-norm

$$\|x\| = \sum_k \left| \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right|^p, \text{ where } 0 < p < 1.$$

One can see from Theorem 2.3 of Jarrah and Malkowsky [5] that the domain μ_T of an infinite matrix $T = (t_{nk})$ in a sequence space μ has a basis if and only if μ has a basis, if T is a triangle. As an immediate consequence of this fact, we derive the following result:

Corollary 1. *Let $0 < p_k \leq H < \infty$ and $\alpha_k = (\widetilde{A}x)_k$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ of the elements of the space $\ell(\widetilde{A}, p)$ by*

$$b_n^{(k)} := \begin{cases} (-1)^{n-k} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad \textit{otherwise} \end{cases} \tag{1.2}$$

for every fixed $k \in \mathbb{N}$. Then, the sequence $\{b^{(k)}\}_{k\in\mathbb{N}}$ given by (1.2) is a basis for the space $\ell(\widetilde{A}, p)$ and any $x \in \ell(\widetilde{A}, p)$ has a unique representation of the form $x := \sum_k \alpha_k b^{(k)}$.

Since the algebraic and topological properties of the space $r^q(p)$ were studied by Altay and Başar in [6], we essentially emphasize the geometric properties of the space $\ell(\widetilde{A}, p)$.

2. The rotundity of the space $\ell(\widetilde{A}, p)$

In this section, we focus on the rotundity and some geometric properties of the space $\ell(\widetilde{A}, p)$. For details, the reader may refer to [7], [8] and [9]. The main purpose of this study is to characterize the rotundity and some other geometric properties of the space $\ell(\widetilde{A}, p)$, the domain of the triangle matrix $\widetilde{A}$ in the sequence space $\ell(p)$.

Definition 2.1. Let $S(X)$ be the unit sphere of a Banach space X. Then a point $x \in S(X)$ is called an extreme point if $2x = y + z$ implies $y = z$ for every y, z $\in$ S(X). A Banach space X is said to be rotund (stricly convex) if every point of $S(X)$ is an extreme point.

Definition 2.2. A Banach space X is said to have the Kadec-Klee property (or property(H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

Definition 2.3. Let X be real vector space. A functional $\sigma : X \to [0, \infty)$ is called a modular if

(i) $\sigma(x) = 0$ if and only if $x = \theta$;
(ii) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$;
(iii) $\sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$

(iv) The modular σ is called convex if $\sigma(\alpha x + \beta y) \leq \alpha\sigma(x) + \beta\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.
A modular σ on X is called
(a) right continuous if $\sigma(\alpha x) \to \sigma(x)$, as $\alpha \to 1^+$ for all $x \in X_\sigma$.
(b) left continuous if $\sigma(\alpha x) \to \sigma(x)$, as $\alpha \to 1^-$ for all $x \in X_\sigma$.
(c) continuous if it is both right and left continuous, where

$$X_\sigma := \left\{ x \in X : \lim_{\alpha \to 0^+} \sigma(\alpha x) = 0 \right\}.$$

We define σ_p on the real sequence space $\ell(\widetilde{A}, p)$ by

$$\sigma_p(x) = \sum_k \left| \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=0}^{k} (\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}) x_j \right|^{p_k}.$$

If $p_k \geq 1$ for all $k \in \mathbb{N}$, by the convexity of the function $t \mapsto |t_k|^{p_k}$ for each $k \in \mathbb{N}$, σ_p is a convex modular on $\ell(\widetilde{A}, p)$.

Proposition 1. *The modular σ_p on $\ell(\widetilde{A}, p)$ satisfies the following properties with $p_k \geq 1$ for all k, we have $M = H$:*

(i) *If $0 < \alpha \leq 1$, then $\alpha^M \sigma_p(x/\alpha) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha\sigma_p(x)$.*
(ii) *If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$.*
(iii) *If $\alpha \geq 1$, then $\sigma_p(x) \geq \alpha\sigma_p(x/\alpha)$.*
(iv) *The modular σ_p is continuous on the space $\ell(\widetilde{A}, p)$.*

Proof. Consider the modular σ_p on $\ell(\widetilde{A}, p)$.

(i) Let $0 < \alpha \leq 1$, then $\alpha^M/\alpha^{p_k} \leq 1$. So, we have

$$\begin{aligned}
\alpha^M \sigma_p\left(\frac{x}{\alpha}\right) &= \alpha^M \sum_k \left|\frac{1}{\alpha}\sum_{j=0}^{k}\frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j\right|^{p_k} \\
&= \alpha^M \sum_k \frac{1}{\alpha^{p_k}}\left|\frac{1}{\lambda_k - \lambda_{k-1}}\sum_{j=0}^{k}(\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j\right|^{p_k} \\
&= \sum_k \frac{\alpha^M}{\alpha^{p_k}}\left|\frac{1}{\lambda_k - \lambda_{k-1}}\sum_{j=0}^{k}(\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j\right|^{p_k} \\
&\leq \sum_k \left|\frac{1}{\lambda_k - \lambda_{k-1}}\sum_{j=0}^{k}(\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j\right|^{p_k} \\
&= \sigma_p(x), \\
\sigma_p(\alpha x) &= \sum_k \left|\frac{\alpha}{\lambda_k - \lambda_{k-1}}\sum_{j=0}^{k}(\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j\right|^{p_k} \\
&= \sum_k \alpha^{p_k}\left|\frac{1}{\lambda_k - \lambda_{k-1}}\sum_{j=0}^{k}(\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j\right|^{p_k} \\
&\leq \alpha\sum_k \left|\frac{1}{\lambda_k - \lambda_{k-1}}\sum_{j=0}^{k}(\lambda_j - 2\lambda_{j-1} + \lambda_{j-2})x_j\right|^{p_k} \\
&= \alpha\sigma_p(x).
\end{aligned}$$

(ii) Let $\alpha \geq 1$. Then, $\alpha^M/\alpha^{p_k} \geq 1$ for all $p_k \geq 1$. So, we have

$$\sigma_p(x) \leq \frac{\alpha^M}{\alpha^{p_k}}\sigma_p(x) = \alpha^M \sigma_p\left(\frac{x}{\alpha}\right).$$

(iii) Let $\alpha \geq 1$. Then, $\alpha/\alpha^{p_k} \leq 1$ for all $p_k \geq 1$. So, we have

$$\sigma_p(x) \geq \frac{\alpha}{\alpha^{p_k}}\sigma_p(x) = \alpha\sigma_p\left(\frac{x}{\alpha}\right).$$

(iv) One can immediately see by Part (ii) for $\alpha > 1$ that

$$\sigma_p(x) \leq \alpha\sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha^M\sigma_p(x). \tag{2.1}$$

By passing to limit as $\alpha \to 1^+$ in (2.1), we have $\sigma_p(\alpha x) \to \sigma_p(x)$. Hence, σ_p is right continuous. If $0 < \alpha < 1$, we have by Part (i) that

$$\alpha^M\sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha\sigma_p(x). \tag{2.2}$$

By letting $\alpha \to 1^-$ in (2.2), we observe that $\sigma_p(\alpha x) \to \sigma_p(x)$. Hence, σ_p is left continuous and so, it is continuous. □

Now, we consider the space $\ell(\widetilde{A},p)$ equipped with the Luxemburg norm given by

$$\|x\| = \inf\left\{\alpha > 0 : \sigma_p\left(\frac{x}{\alpha}\right) \le 1\right\}.$$

Proposition 2. *For any $x \in \ell(\widetilde{A},p)$, the following statements hold:*

(i) *If $\|x\| < 1$, then $\sigma_p(x) \le \|x\|$.*
(ii) *If $\|x\| > 1$, then $\sigma_p(x) \ge \|x\|$.*
(iii) *$\|x\| = 1$ if and only if $\sigma_p(x) = 1$.*
(iv) *$\|x\| < 1$ if and only if $\sigma_p(x) < 1$.*
(v) *$\|x\| > 1$ if and only if $\sigma_p(x) > 1$.*

Proof. Let $x \in \ell(\widetilde{A},p)$.

(i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$. By the definition of $\|\cdot\|$, there exists an $\alpha > 0$ such that $\|x\| + \varepsilon > \alpha$ and $\sigma_p(x) \le 1$. From Parts (i) and (ii) of Proposition 1, we obtain

$$\sigma_p(x) \le \sigma_p\left[(\|x\|+\varepsilon)\frac{x}{\alpha}\right] \le (\|x\|+\varepsilon)\sigma_p\left(\frac{x}{\alpha}\right) \le \|x\| + \varepsilon.$$

Since ε is arbitrary, we have (i).

(ii) If we choose $\varepsilon > 0$ such that $0 < \varepsilon < 1-(1/\|x\|)$, then $1 < (1-\varepsilon)\|x\| < \|x\|$. By the definition of $\|\cdot\|$ and Part (i) of Proposition 1, we have

$$1 < \sigma_p\left[\frac{x}{(1-\varepsilon)\|x\|}\right] \le \frac{1}{(1-\varepsilon)\|x\|}\sigma_p(x).$$

So $(1-\varepsilon)\|x\| < \sigma_p(x)$ for all $\varepsilon \in (0, 1-(1/\|x\|))$. This implies that $\|x\| < \sigma_p(x)$.

(iii) Since σ_p is continuous, we directly have (iii).
(iv) This follows from Parts (i) and (iii).
(v) This follows from Parts (ii) and (iii).

□

Theorem 2.4. *$\ell(\widetilde{A},p)$ is a Banach space with the Luxemburg norm.*

Proof. Let $S_x = \{\alpha > 0 : \sigma_p(x/\alpha) \le 1\}$ and $\|x\| = \inf S_x$ for all $x \in \ell(\widetilde{A},p)$. Then, $S_x \subset (0,\infty)$. Therefore, $\|x\| \ge 0$ for all $x \in \ell(\widetilde{A},p)$.

For $x = \theta$, $\sigma_p(\theta) = 0$ for all $\alpha > 0$. Hence, $S_0 = (0,\infty)$ and $\|\theta\| = \inf S_0 = \inf(0,\infty) = 0$.

Let $x \ne \theta$ and $Y = \{kx : k \in \mathbb{C} \text{ and } x \in \ell(\widetilde{A},p)\}$ be a non-empty subset of $\ell(\widetilde{A},p)$. Since $Y \subsetneq S[\ell(\widetilde{A},p)]$, there exists $k_1 \in \mathbb{C}$ such that $k_1 x \notin S[\ell(\widetilde{A},p)]$. Obviously $k_1 \ne 0$. We assume that $0 < \alpha < 1/k_1$ and $\alpha \in S_x$. Then, $(x/\alpha) \in S[\ell(\widetilde{A},p)]$. Since $|k_1\alpha| < 1$, we get

$$k_1 x = k_1 \alpha \frac{x}{\alpha} \in S[\ell(\widetilde{A},p)]$$

which contradicts the assumption. Hence, we obtain that if $\alpha \in S_x$, then $\alpha > 1/|k_1|$. This means that $\|x\| \geq 1/|k_1| > 0$. Thus, we conclude that $\|x\| = 0$ if and only if $x = \theta$.

Now, let $k \neq 0$ and $\alpha \in S_{kx}$. Then, we have

$$\sigma_p\left(\frac{kx}{\alpha}\right) \leq 1 \quad and \quad \frac{kx}{\alpha} \in S[\ell(\widetilde{A}, p)].$$

Therefore, we obtain

$$\frac{|k|x}{\alpha} = \frac{|k|}{k} \times \frac{kx}{\alpha} \in S[\ell(\widetilde{A}, p)] \quad and \quad \frac{\alpha}{|k|} \in S_x.$$

That is, $\|x\| \leq \alpha/|k|$ and $|k|\|x\| \leq \alpha$ for all $\alpha \in S_{kx}$. So, $|k|\|x\| \leq \|kx\|$. If we take $1/k$ and kx instead of k and x, respectively, then we obtain that

$$\left|\frac{1}{k}\right| \|kx\| \leq \left\|\frac{1}{k}kx\right\| = \|x\| \quad and \quad \|kx\| \leq |k|\|x\|.$$

Hence, we see $\|kx\| = |k|\|x\|$ which also holds when $k = 0$.

To prove the triangle inequality, let x, $y \in S[\ell(\widetilde{A}, p)]$ and $\varepsilon > 0$ be given. Then, there exist $\alpha \in S_x$ and $\beta \in S_y$ such that $\alpha < \|x\| + \varepsilon$ and $\beta < \|y\| + \varepsilon$. Since $S[\ell(\widetilde{A}, p)]$ is convex,

$$\frac{x}{\alpha} \in S[\ell(\widetilde{A}, p)],\ \frac{y}{\beta} \in S[\ell(\widetilde{A}, p)],\ \frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}\left(\frac{x}{\alpha}\right) + \frac{\beta}{\alpha+\beta}\left(\frac{y}{\beta}\right) \in S[\ell(\widetilde{A}, p)].$$

Therefore, $\alpha + \beta \in S_{x+y}$. Then, we have $\|x + y\| \leq \alpha + \beta < \|x\| + \|y\| + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain $\|x + y\| \leq \|x\| + \|y\|$. Hence, $\|x\| = \inf\{\alpha > 0 : \sigma_p(x/\alpha) \leq 1\}$ is a norm on $\ell(\widetilde{A}, p)$.

Now, we show that every Cauchy sequence in $\ell(\widetilde{A}, p)$ is convergent with respect to the Luxemburg norm. Let $\left\{x_k^{(n)}\right\}$ be a Cauchy sequence in $\ell(\widetilde{A}, p)$ and $\varepsilon \in (0, 1)$. Thus, there exists n_0 such that $\|x^{(n)} - x^{(m)}\| < \varepsilon$ for all $n, m \geq n_0$. By Part (i) of Proposition 2, we have

$$\sigma_p\left(x^{(n)} - x^{(m)}\right) \leq \left\|x^{(n)} - x^{(m)}\right\| < \varepsilon \tag{2.3}$$

for all $n, m \geq n_0$. This implies that

$$\sum_k \left|\left[\widetilde{A}\left(x^{(n)} - x^{(m)}\right)\right]_k\right|^{p_k} < \varepsilon. \tag{2.4}$$

Then, for each fixed k and for all $n, m \geq n_0$,

$$\left|\left[\widetilde{A}\left(x^{(n)} - x^{(m)}\right)\right]_k\right|^{p_k} = \left|\left(\widetilde{A}x^{(n)}\right)_k - \left(\widetilde{A}x^{(m)}\right)_k\right| < \varepsilon.$$

Hence, the sequence $\{(\widetilde{A}x^{(n)})_k\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there is $(\widetilde{A}x)_k \in \mathbb{R}$ such that $\left(\widetilde{A}x^{(m)}\right)_k \to (\widetilde{A}x)_k$, as $m \to \infty$. Therefore, as $m \to \infty$

by (2.4) we have

$$\sum_k \left| \left[\widetilde{A}\left(x^{(n)} - x\right) \right]_k \right|^{p_k} < \varepsilon$$

for all $n \geq n_0$.

Now, we have to show that (x_k) is an element of $\ell(\widetilde{A}, p)$. Since $\left(\widetilde{A}x^{(m)}\right)_k \to (\widetilde{A}x)_k$, as $m \to \infty$, we have

$$\lim_{m\to\infty} \sigma_p\left(x^{(n)} - x^{(m)}\right) = \sigma_p\left(x^{(n)} - x\right). \tag{2.5}$$

Then, we see by (2.3) that $\sigma_p\left(x^{(n)} - x\right) \leq \|x^{(n)} - x\| < \varepsilon$ for all $n \geq n_0$. This implies that $x^{(n)} \to x$, as $n \to \infty$. So, we have $x = x^{(n)} - \left(x^{(n)} - x\right) \in \ell(\widetilde{A}, p)$. Therefore, the sequence space $\ell(\widetilde{A}, p)$ is complete with respect to Luxemburg norm. This completes the proof. □

Theorem 2.5. *The space $\ell(\widetilde{A}, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Let $\ell(\widetilde{A}, p)$ be rotund and choose $k \in \mathbb{N}$ such that $p_k = 1$. Consider the following sequences given by

$$x = \left(1, \frac{-\lambda_0}{\lambda_1 - 2\lambda_0}, 0, 0, \dots\right) \quad \text{and} \quad y = \left(0, \frac{\lambda_1 - \lambda_0}{\lambda_1 - 2\lambda_0}, -\frac{\lambda_1 - \lambda_0}{\lambda_2 - 2\lambda_1 + \lambda_0}, 0, 0, \dots\right).$$

Then, obviously $x \neq y$ and $\sigma_p(x) = \sigma_p(y) = \sigma_p\left(\frac{x+y}{2}\right) = 1$. By Part (iii) of Proposition 2, x, y, $(x+y)/2 \in S[\ell(\widetilde{A}, p)]$ which leads us to the contradiction that the sequence space $\ell(\widetilde{A}, p)$ is not rotund. Hence, $p_k > 1$ for all $k \in \mathbb{N}$. Conversely, let $x \in S[\ell(\widetilde{A}, p)]$ and $v, z \in S[\ell(\widetilde{A}, p)]$ with $x = (v+z)/2$. By convexity of σ_p and Part (iii) of Proposition 2, we have

$$1 = \sigma_p(x) \leq \frac{\sigma_p(v) + \sigma_p(z)}{2} \leq \frac{1}{2} + \frac{1}{2} = 1$$

which gives that $\sigma_p(v) = \sigma_p(z) = 1$ and

$$\sigma_p(x) = \sigma_p((v+z)/2) = \frac{\sigma_p(v) + \sigma_p(z)}{2}. \tag{2.6}$$

Also, we obtain from (2.6) that

$$\left| \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=0}^{k} \lambda_i \frac{(v_j + z_j)}{2} - 2\lambda_{j-1} \frac{(v_j + z_j)}{2} + \lambda_{j-2} \frac{(v_j + z_j)}{2} \right|^{p_k} \tag{2.7}$$
$$= \frac{1}{2} \left| \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} v_j \right|^{p_k} + \frac{1}{2} \left| \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} z_j \right|^{p_k}$$

for all $k \in \mathbb{N}$. Since the function $t \mapsto |t|_k^p$ is strictly convex for all $k \in \mathbb{N}$, it follows by (2.7) that $v_k = z_k$ for all k $\in \mathbb{N}$. Hence, $v = z$. That is, the sequence space $\ell(\widetilde{A}, p)$ is rotund. □

Theorem 2.6. *Let $x \in \ell(\widetilde{A}, p)$. Then, the following statements hold:*

(i) $0 < \alpha < 1$ *and* $\|x\| > \alpha$ *imply* $\sigma_p(x) > \alpha^M$.
(ii) $\alpha \geq 1$ *and* $\|x\| < \alpha$ *imply* $\sigma_p(x) < \alpha^M$.

Proof. Let $x \in \ell(\widetilde{A}, p)$.

(i) Suppose that $\|x\| > \alpha$ with $0 < \alpha < 1$. Then, $\|x/\alpha\| > 1$. By Part (ii) of Proposition 2, $\|x/\alpha\| > 1$ implies $\sigma_p(x/\alpha) \geq \|x/\alpha\| > 1$. That is, $\sigma_p(x/\alpha) > 1$. Since $0 < \alpha < 1$, by Part (i) of Proposition 1, we get $\alpha^M \sigma_p(x/\alpha) \leq \sigma_p(x)$. Thus, we have $\alpha^M < \sigma_p(x)$.
(ii) Let $\|x\| < \alpha$ with $\alpha \geq 1$. Then $\|x/\alpha\| < 1$. By Part (i) of Proposition 2, $\|x/\alpha\| < 1$ implies $\sigma_p(x/\alpha) \leq \|x/\alpha\| < 1$. That is, $\sigma_p(x/\alpha) < 1$. If $\alpha = 1$, then $\sigma_p(x/\alpha) = \sigma_p(x) < 1 = \alpha^M$. If $\alpha > 1$, then by Part (ii) of Proposition 1, we have $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$. This means that $\sigma_p(x) < \alpha^M$.

□

Theorem 2.7. *Let (x_n) be a sequence in $\ell(\widetilde{A}, p)$. Then, the following statements hold:*

(i) $\|x_n\| \to 1$, *as* $n \to \infty$ *implies* $\sigma_p(x_n) \to 1$, *as* $n \to \infty$.
(ii) $\sigma_p(x_n) \to 0$, *as* $n \to \infty$ *implies* $\|x_n\| \to 0$, *as* $n \to \infty$.

Proof. Let (x_n) be a sequence in $\ell(\widetilde{A}, p)$.

(i) $\|x_n\| \to 1$, as $n \to \infty$ and $\varepsilon \in (0, 1)$. Then, there exists $n_0 \in \mathbb{N}$ such that $1 - \varepsilon < \|x_n\| < \varepsilon + 1$ for all $n \geq n_0$. By Parts (i) and (ii) of Theorem 2.6, $1 - \varepsilon < \|x_n\|$ implies $\sigma_p(x_n) > (1 - \varepsilon)^M$ and $\|x_n\| < \varepsilon + 1$ implies $\sigma_p(x_n) < (1 + \varepsilon)^M$ for all $n \geq n_0$. This means $\varepsilon \in (0, 1)$ and for all $n \geq n_0$ there exists $n_0 \in \mathbb{N}$ such that $(1 - \varepsilon)^M < \sigma_p(x_n) < (1 + \varepsilon)^M$ for all $n \geq n_0$. That is, $\sigma_p(x_n) \to 1$, as $n \to \infty$.
(ii) We assume that $\|x_n\| \nrightarrow 0$, as $n \to \infty$ and $\varepsilon \in (0, 1)$. Then, there exists a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \varepsilon$ for all $k \in \mathbb{N}$. By Part (i) of Theorem 2.6, $0 < \varepsilon < 1$ and $\|x_{n_k}\| > \varepsilon$ imply $\sigma_p(x_{n_k}) > \varepsilon^M$. Thus, $\sigma_p(x_n) \nrightarrow 0$, as $n \to \infty$. Hence, we obtain that $\sigma_p(x_n) \to 0$, as $n \to \infty$ implies $\|x_n\| \to 0$, as $n \to \infty$.

□

Theorem 2.8. *Let $x \in \ell(\widetilde{A}, p)$ and $\left(x^{(n)}\right) \subset \ell(\widetilde{A}, p)$. If $\sigma_p\left(x^{(n)}\right) \to \sigma_p(x)$, as $n \to \infty$ and $x_k^{(n)} \to x_k$, as $n \to \infty$ for all $k \in \mathbb{N}$, then $x^{(n)} \to x$, as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be given. Since $\sigma_p(x) = \sum_k \left|(\widetilde{A}x)_k\right|^{p_k} < \infty$, $x \in \ell(\widetilde{A}, p)$ there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \left|(\widetilde{A}x)_k\right|^{p_k} < \frac{\varepsilon}{3(2^{M+1})}. \tag{2.8}$$

It follows from the equality

$$\lim_{n\to\infty}\left[\sigma_p(x^{(n)}) - \sum_{k=0}^{k_0}\left|\left(\widetilde{A}x^{(n)}\right)_k\right|^{p_k}\right] = \sigma_p(x) - \sum_{k=0}^{k_0}\left|\left(\widetilde{A}x\right)_k\right|^{p_k}$$

that there exists $n_0 \in \mathbb{N}$ and for all $k \in \mathbb{N}$

$$\sigma_p(x^{(n)}) - \sum_{k=0}^{k_0}\left|\left(\widetilde{A}x^{(n)}\right)_k\right|^{p_k} < \sigma_p(x) - \sum_{k=0}^{k_0}\left|\left(\widetilde{A}x\right)_k\right|^{p_k} + \frac{\varepsilon}{3(2^M)} \tag{2.9}$$

and for all $k \in \mathbb{N}$

$$\sum_{k=0}^{k_0}\left|\left(\widetilde{A}(x^{(n)} - x)\right)_k\right|^{p_k} < \frac{\varepsilon}{3}. \tag{2.10}$$

Therefore, we obtain from (2.8), (2.9) and (2.10) that

$$\begin{aligned}
\sigma_p(x_n - x) &= \sum_{k=0}^{\infty}\left|\{\widetilde{A}(x^{(n)} - x)\}_k\right|^{p_k} \\
&< \sum_{k=0}^{k_0}\left|\{\widetilde{A}(x^{(n)} - x)\}_k\right|^{p_k} + \sum_{k=k_0+1}^{\infty}\left|\{\widetilde{A}(x^{(n)} - x)\}_k\right|^{p_k} \\
&< \frac{\varepsilon}{3} + 2^M\left[\sum_{k=k_0+1}^{\infty}\left|(\widetilde{A}x^{(n)})_k\right|^{p_k} + \sum_{k=k_0+1}^{\infty}\left|(\widetilde{A}x)_k\right|^{p_k}\right] \\
&< \frac{\varepsilon}{3} + 2^M\left[\sigma_p(x^{(n)}) - \sum_{k=0}^{k_0}\left|(\widetilde{A}x^{(n)})_k\right|^{p_k} + \sum_{k=k_0+1}^{\infty}\left|(\widetilde{A}x)_k\right|^{p_k}\right] \\
&< \frac{\varepsilon}{3} + 2^M\left[\sigma_p(x) - \sum_{k=0}^{k_0}\left|(\widetilde{A}x)_k\right|^{p_k} + \frac{\varepsilon}{3(2^M)} + \sum_{k=k_0+1}^{\infty}\left|(\widetilde{A}x)_k\right|^{p_k}\right] \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2^M\left[2\sum_{k=k_0+1}^{\infty}\left|(\widetilde{A}x)_k\right|^{p_k}\right] \\
&< \frac{2\varepsilon}{3} + \frac{2^{M+1}\epsilon}{3(2^{M+1})} = \varepsilon.
\end{aligned}$$

This means that $\sigma_p\left(x^{(n)}-x\right) \to 0$, as $n \to \infty$. By Part (i) of Theorem 2.7, $\sigma_p\left(x^{(n)}-x\right) \to 0$, as $n \to \infty$ implies $\|x_n - x\| \to 0$, as $n \to \infty$. Hence, $x_n \to x$, as $n \to \infty$. ☐

Theorem 2.9. *The sequence space $\ell(\widetilde{A},p)$ has the Kadec-Klee property.*

Proof. Let $x \in S\left[\ell(\widetilde{A},p)\right]$ and $\left(x^{(n)}\right) \subset \ell(\widetilde{A},p)$ such that $\|x^{(n)}\| \to 1$ and $x^{(n)} \xrightarrow{w} x$ be given. By Part (ii) of Theorem 2.7, we have $\sigma_p\left(x^{(n)}\right) \to 1$ as $n \to \infty$. Also $x \in S\left[\ell(\widetilde{A},p)\right]$ implies $\|x\| = 1$. By Part (iii) of Proposition 2, we obtain $\sigma_p(x) = 1$. Therefore, we have $\sigma_p(x^{(n)}) \to \sigma_p(x)$, as $n \to \infty$.

Since $x^{(n)} \xrightarrow{w} x$, as $n \to \infty$ and $q_k : \ell(\widetilde{A},p) \to \mathbb{R}$ defined by $q_k(x) = x_k$ is continuous, $x_k^{(n)} \to x_k$, as $n \to \infty$ for all $k \in \mathbb{N}$. Therefore, $x^{(n)} \to x$, as $n \to \infty$.

Because of any weakly convergent sequence in $\ell(\widetilde{A},p)$ is convergent, the sequence space $\ell(\widetilde{A},p)$ has the Kadec-Klee property. ☐

Conclusion

Let $0 < r < 1$, $q = (q_k)$ be a sequence of non-negative reals with $q_0 > 0$ and $Q_n = \sum_{k=0}^{n} q_k$ for all $n \in \mathbb{N}$, $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ be the convergent sequences. Suppose that the sequences $u = (u_k)$ and $v = (v_k)$ consist of non-zero entries; $u, s \in \mathbb{R}$, and $\lambda = (\lambda_n)$ be the strictly increasing sequence of positive real numbers tending to infinity with $\lambda_{n+1} \geq 2\lambda_n$.

Let us define the Riesz matrix $R^q = (r_{nk}^q)$ with respect to the sequence $q = (q_k)$, the double band matrix $F = (f_{nk})$ defined by the sequence (f_n) of Fibonacci numbers, the matrix $A^r = (a_{nk}^r)$, the generalized difference matrix $B(u,s) = \{b_{nk}(u,s)\}$, the matrix $A^u = (a_{nk}^u)$, the double sequential band matrix $B(\widetilde{r},\widetilde{s}) = \{b_{nk}(r_k,s_k)\}$, the matrix $\widetilde{A} = \{a_{nk}(\lambda)\}$ and the Nörlund matrix $N^q = (a_{nk}^q)$ with respect to the sequence $q = (q_k)$ by

$$r_{nk}^q := \begin{cases} \frac{q_k}{Q_n} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases} \qquad f_{nk} := \begin{cases} -\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ \frac{f_n}{f_{n+1}} & , \quad k = n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

$$a_{nk}^r := \begin{cases} \frac{1+r^k}{n+1} u_k & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases} \qquad b_{nk}(u,s) := \begin{cases} u & , \quad k = n, \\ s & , \quad k = n-1, \\ 0 & , \quad 0 \leq k < n-1 \quad \text{or} \quad k > n, \end{cases}$$

$$a_{nk}^u := \begin{cases} (-1)^{n-k} u_k & , \quad n-1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n-1 \quad \text{or} \quad k > n, \end{cases}$$

$$b_{nk}(r_k, s_k) = \begin{cases} r_k & , \quad k = n, \\ s_k & , \quad k = n-1, \\ 0 & , \quad 0 \leq k < n-1 \quad \text{or} \quad k > n, \end{cases}$$

$$a_{nk}(\lambda) := \begin{cases} \frac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_n - \lambda_{n-1}} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases} \qquad a_{nk}^q = \begin{cases} \frac{q_{n-k}}{Q_n} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$.

For concerning literature about the geometric properties of the domain of the infinite matrix A in the sequence space $\ell(p)$, the following table may be useful:

A	the space λ	geometric properties of λ_A	refer to:
A^r	$\ell(p)$	$a^r(u,p)$	[10]
$B(u,s)$	$\ell(p)$	$\widehat{\ell}(p)$	[11]
A^u	$\ell(p)$	$bv(u,p)$	[12]
$B(\widetilde{r},\widetilde{s})$	$\ell(p)$	$\ell(\widetilde{B},p)$	[13, 14]
F	$\ell(p)$	$\ell(F,p)$	[15]
N^q	$\ell(p)$	$N^q(p)$	[16]

Table 1: The domains of some triangle matrices in the spaces $\ell(p)$.

In the special case $q_k = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2}$ and $Q_n = \lambda_n - \lambda_{n-1}$, R^q is reduced to $\widetilde{A}$. So, the space $\ell(\widetilde{A},p)$ can be seen as a special case of the space $r^q(p)$, the domain of the Riesz mean R^q in the Maddox' space $\ell(p)$ introduced by Altay and Başar [6]. Since the geometric properties of the space $r^q(p)$ was not investigated the main results of the present paper are not contained in Altay and Başar [6]. So, the main results of the present study can be seen as the complementary results for Altay and Başar [6].

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