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ON SURFACE THEORY IN 3-DIMENSIONAL ALMOST CONTACT METRIC MANIFOLD

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ABSTRACT. In this paper, we study surface theory in 3-dimensional almost contact metric manifolds by using cross product defined by Camcı [6] . Camcı also studied the theory of curves using the new cross product on this manifolds. In this study, we have defined unit normal vector field of any surface in $\mathbb{R}^3(-3)$ and then, we investigate shape operator matrix of the surface. Morever, we calculate the formulas of Gaussian and mean curvatures of a surface in $\mathbb{R}^3(-3)$.

1. Introduction

In contact geometry, a lot of studies have been published about curves such as legendre curves and finite type curves ([1, 2, 3, 4, 5]). Particularly, the Legendre curves are very important in the studies of contact manifolds where a diffeomorphism is a contact transformation if and only if any Legendre curves in a domain of it go to Legendre curves. Morever, in a 3-dimensional Sasakian manifold, the Legendre curves are studied by Baikoussis and Blair who gave the Frenet 3-frame in this space ([3]). Then, Camci has studied the curves theory in contact geometry for any curves ([4]).

But, few studies have been published the surface theory in contact geometry since Camci defined a new cross product in 3-dimensional almost contact metric manifold and studied the theory of curves using this new cross product in this manifold ([6]). And then, Gök has studied the surface theory in 3-dimensional almost contact metric manifold by using cross product defined by Camci ([8]).

In this paper, we study surface theory in 3-dimensional almost contact metric manifold by using cross product defined by Camci ([6]) and we define unit normal vector of any surface in $\mathbb{R}^3(-3)$ and then, we investigate shape operator matrix of the surface. Morever, we calculate the formulas of Gaussian and mean curvature using the new cross product in this manifold.

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2. Preliminaries

Let M be a $(2n+1)$ dimensional differentiable manifold which has a 1-form η, such that

$$\eta \wedge (d\eta)^n \neq 0$$

on M. In this case, M is called contact manifold and η is called a contact 1-form. There exists a unique ξ, called characteristic vector field of η, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for all $X \in \chi(M)$. D is said to be contact distribution defined by

$$D = \{x \in \chi(M) : \eta(X) = 0\}.$$

(φ, ξ, η) is called an almost contact structure on M^{2n+1} where φ, ξ, η are type $(1,1), (0,1)$ and $(1,0)$ tensor field, respectively, satisfying the equations

$$\varphi^2(X) = -X + \eta(X)\xi \quad , \quad \varphi(\xi) = 0 \quad , \quad \eta(\xi) = 1 \text{ and } \eta \circ \varphi = 0$$

where the endomorphism φ has rank $2n$.

(φ, ξ, η, g) is called an almost contact metric structure on M^{2n+1} where g is a Riemannian metric, satisfying

$$\begin{gathered} g(\varphi(X), \varphi(Y)) = g(X,Y) - \eta(X)\eta(Y), \\ g(X, \varphi(Y)) = d\eta(X,Y), \\ \eta(X) = g(X, \xi) \end{gathered}$$

for all $X, Y \in \chi(M)$.

Let M be a $(2n+1)$-dimensional manifold which is called Sasaki manifold if it is endowed with a normal contact metric structure (φ, ξ, η, g). We know that an almost contact metric structure on M is sasakian structure if and only if

$$(\nabla_X \varphi)(Y) = g(X,Y)\xi - \eta(Y)X$$

for all $X, Y \in \chi(M)$, where ∇ is the Riemannian connection of g.

Let (x, y, z) be the standart coordinates on $\mathbb{R}^3$. Let consider the 1-form

$$\eta = \frac{1}{2}(dz - ydx)$$

on $\mathbb{R}^3$ and $\xi = 2\frac{\partial}{\partial z}$ on $\mathbb{R}^3$, then we can easily see that ξ is a characteristic vector field.

If the Riemannian metric is defined by

$$g = \frac{1}{4}(dx^2 + dy^2) + \eta \otimes \eta$$

and the endomorphism of φ is defined by the matrix

$$\varphi = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{bmatrix},$$

then we know that (φ, ξ, η, g) is a Sasakian structure and the sectional curvature φ of this space is equal to -3. So, it is defined by $\mathbb{R}^3(-3)$. It is well known that

$$\psi = \left\{ e = e_1 = 2\frac{\partial}{\partial y}, \, \varphi(e) = e_2 = 2\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \, \xi = e_3 = 2\frac{\partial}{\partial z} \right\} \tag{2.1}$$

is an orthonormal basis with respect to g in $\mathbb{R}^3(-3)$. Let $X = x_1 e + x_2\varphi(e) + x_3\xi$ and $Y = y_1 e + y_2\varphi(e) + y_3\xi$ be vector fields in $\mathbb{R}^3(-3)$, then we can easily see that $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$ is a 3-dimensional almost contact metric manifold and φ and η satisfying the equations

$$\begin{aligned} \varphi(X) &= -x_2 e + x_1\varphi(e), \\ \varphi(Y) &= -y_2 e + y_1\varphi(e), \\ \eta(X) &= x_3. \end{aligned}$$

In a 3-dimensional almost contact metric manifold, Camci stated the following definition and theorem.

Definition 2.1. Let $M^3 = (M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. The cross product $\wedge : \chi(M) \times \chi(M) \longrightarrow \chi(M)$ is defined by

$$X \wedge Y = -g(X, \varphi(Y))\xi - \eta(Y)\varphi(X) + \eta(X)\varphi(Y) \tag{2.2}$$

where X, $Y \in \chi(M)$ ([6]).

Theorem 2.2. *Let $M^3 = (M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. Then, for all X, Y, $Z \in \chi(M)$ the cross product satisfying the following properties:*

i) The cross product is bilinear and antisymmetric.
ii) $X \wedge Y$ is perpendicular both of X and Y.
iii)

$$Y \wedge \varphi(X) = g(X, Y)\xi - \eta(Y)X, \tag{2.3}$$

$$\varphi(X) = \xi \wedge X. \tag{2.4}$$

iv) Define a mixed product by

$$\begin{aligned} (X, Y, Z) &= g(X \wedge Y, Z) \\ &= -\left[g(X, \varphi(Y))\eta(Z) + g(Y, \varphi(Z))\eta(X) + g(Z, \varphi(X))\eta(Y)\right] \end{aligned} \tag{2.5}$$

and

$$(X, Y, Z) = (Y, Z, X) = (Z, X, Y).$$

v)

$$\left\{ \begin{array}{c} g(X, \varphi(Y))Z + g(Y, \varphi(Z))X + g(Z, \varphi(X))Y = -\det(X, Y, Z)\xi, \\ (X \wedge Y) \wedge Z = g(X, Z)Y - g(Y, Z)X, \\ (X \wedge Y) \wedge Z + (Y \wedge Z) \wedge X + (Z \wedge X) \wedge Y = 0. \end{array} \right. \tag{2.6}$$

vi)

$$\begin{cases} g(X\wedge Y, Z\wedge W) = g(X,Z)g(Y,W) - g(Y,Z)g(X,W), \\ g(X\wedge Y, X\wedge Y) = \|X\wedge Y\|^2 = g(X,X)g(Y,Y) - g^2(X,Y), \end{cases} \tag{2.7}$$

for the proofs of the above equalities (see [6, 8]).

3. Shape operator matrix of a surface in 3-dimensional almost contact metric manifold

In this section, we first recall the definition of a shape operator in general mean and then we investigate shape operator matrix of a surface, the formulas of Gaussian and mean curvatures in the 3-dimensional almost contact metric manifold using its unit normal vector field.

Definition 3.1. Let M be a surface in E^n. The linear map $S : \chi(M) \to \chi(M)$ defined by

$$S(X) := D_X N \ , \ X \in \chi(M),$$

is called the shape operator on M, where D is the Riemannian connection in $\mathbb{E}^n$ and N is the unit normal vector field of the surface M.

Proposition 1. *Let U denote an open set in the plane $\mathbb{R}^2$. The open set U will typically be an open disk or open rectangle. Let*

$$\begin{aligned} X \quad &: \quad U \longrightarrow \mathbb{R}^3(-3) \\ &: \quad (u,v) \longmapsto X(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v)) \end{aligned}$$

be a parameterization at a point $P \in M$ of a surface M in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$. Tangent vectors for the u and $v-$parameter curves are given by differentiating of the $f_i(u,v)$. According to the basis $\{e, \varphi(e), \xi\}$ of $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$, we can write

$$X_u = \frac{1}{2} f_{2,u} e + \frac{1}{2} f_{1,u} \varphi(e) + \frac{1}{2}(f_{3,u} - f_2 f_{1,u})\xi, \tag{3.1}$$

$$X_v = \frac{1}{2} f_{2,v} e + \frac{1}{2} f_{1,v} \varphi(e) + \frac{1}{2}(f_{3,v} - f_2 f_{1,v})\xi. \tag{3.2}$$

where $f_{i,u}$ and $f_{i,v}$ $(1 \le i \le 3)$ mean that the first derivatives of $f_i(u,v)$ according to the u and $v-$parameters .

Proof. Tangent vector of the $u-$parameter curve on a surface $M : X(u,v)$ in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$ is

$$X_u = f_{1,u} \frac{\partial}{\partial_x} + f_{2,u} \frac{\partial}{\partial_y} + f_{3,u} \frac{\partial}{\partial_z},$$

from the equation (2.1) we have

$$X_u = \frac{1}{2} f_{2,u} e + \frac{1}{2} f_{1,u} \varphi(e) + \frac{1}{2}(f_{3,u} - f_2 f_{1,u})\xi$$

and similarly

$$X_v = \frac{1}{2} f_{2,v} e + \frac{1}{2} f_{1,v} \varphi(e) + \frac{1}{2}(f_{3,v} - f_2 f_{1,v})\xi,$$

which complete the proof. □

Theorem 3.2. *Let U denote an open set in the plane $\mathbb{R}^2$ and*

$$\begin{aligned} X \quad &: \quad U \longrightarrow \mathbb{R}^3(-3) \\ &: \quad (u,v) \longmapsto X(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v)) \end{aligned}$$

be a parameterization at a point. $P \in M$ of a surface M in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$. The unit normal vector field of M in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$ is

$$\begin{aligned} N \quad = \quad & \frac{1}{4\sqrt{EG - F^2}} \left[f_{1,u}(f_{3,v} - f_2 f_{1,v}) - f_{1,v}(f_{3,u} - f_2 f_{1,u}) \right] e \\ & + \frac{1}{4\sqrt{EG - F^2}} \left[f_{2,v}(f_{3,u} - f_2 f_{1,u}) - f_{2,u}(f_{3,v} - f_2 f_{1,v}) \right] \varphi(e) \\ & + \frac{1}{4\sqrt{EG - F^2}} (f_{1,v} f_{2,u} - f_{1,u} f_{2,v})\xi \end{aligned} \tag{3.3}$$

where

$$E = \frac{1}{4} f_{2,u}^2 + \frac{1}{4} f_{1,u}^2 + \frac{1}{4}(f_{3,u} - f_2 f_{1,u})^2, \tag{3.4}$$

$$G = \frac{1}{4} f_{2,v}^2 + \frac{1}{4} f_{1,v}^2 + \frac{1}{4}(f_{3,v} - f_2 f_{1,v})^2, \tag{3.5}$$

$$F = \frac{1}{4} f_{2,u} f_{2,v} + \frac{1}{4} f_{1,u} f_{1,v} + \frac{1}{4}(f_{3,u} - f_2 f_{1,u})(f_{3,v} - f_2 f_{1,v}). \tag{3.6}$$

Proof. From the Definition (2.1), we know

$$X_u \wedge X_v = -g(X_u, \varphi(X_v))\xi - \eta(X_v)\varphi(X_u) + \eta(X_u)\varphi(X_v).$$

By using the Proposition (1) and following equations

$$\varphi(X_u) = -\frac{1}{2} f_{1,u} e + \frac{1}{2} f_{2,u} \varphi(e) \quad , \quad \varphi(\ X_v) = -\frac{1}{2} f_{1,v} e + \frac{1}{2} f_{2,v} \varphi(e), \tag{3.7}$$

$$\eta(X_u) = \frac{1}{2}(f_{3,u} - f_2 f_{1,u}) \quad , \quad \eta(X_v) = \frac{1}{2}(f_{3,v} - f_2 f_{1,v}) \tag{3.8}$$

we have

$$\begin{aligned} X_u \wedge X_v \quad = \quad & \frac{1}{4} \left[f_{1,u}(f_{3,v} - f_2 f_{1,v}) - f_{1,v}(f_{3,u} - f_2 f_{1,u}) \right] e \\ & + \frac{1}{4} \left[f_{2,v}(f_{3,u} - f_2 f_{1,u}) - f_{2,u}(f_{3,v} - f_2 f_{1,v}) \right] \varphi(e) \\ & + \frac{1}{4} (f_{1,v} f_{2,u} - f_{1,u} f_{2,v})\xi. \end{aligned} \tag{3.9}$$

Then, via the Theorem (2.2) the norm of $X_u \wedge X_v$ is given by

$$\|X_u \wedge X_v\| = \left(g(X_u, X_u) g(X_v, X_v) - g^2(X_u, X_v) \right)^{\frac{1}{2}}$$

where $g(X_u, X_u) = E$, $g(X_u, X_v) = F$ and $g(X_v, X_v) = G$.

Since $N = \dfrac{X_u \wedge X_v}{\|X_u \wedge X_v\|}$, we have

$$N = \frac{1}{4\sqrt{EG - F^2}} \begin{pmatrix} [f_{1,u}(f_{3,v} - f_2 f_{1,v}) - f_{1,v}(f_{3,u} - f_2 f_{1,u})]\, e \\ +[f_{2,v}(f_{3,u} - f_2 f_{1,u}) - f_{2,u}(f_{3,v} - f_2 f_{1,v})]\, \varphi(e) \\ +(f_{1,v} f_{2,u} - f_{1,u} f_{2,v})\xi \end{pmatrix},$$

which completes the proof. □

Remark 3.3. Let $M : X(u,v)$ be a surface in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$. We know that all u and $v-$ parameter curves are lines of curvature if and only if $F = 0$ and $m = 0$. So, we consider that they are not lines of curvature. Because, it can easily convert to preceding case.

Definition 3.4. Let $X = x_1 e + x_2 \varphi(e) + x_3 \xi$ and $Y = y_1 e + y_2 \varphi(e) + y_3 \xi$ be a differentiable vector fields in an open set $U \subset M$ of a regular surface M in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$. By using the Christoffel symbols on M in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$, we have

$$\begin{aligned} \nabla_e \varphi(e) &= \xi = -\nabla_{\varphi(e)} e, \\ \nabla_\xi e &= -\varphi(e) = \nabla_e \xi, \\ \nabla_\xi \varphi(e) &= e = \nabla_{\varphi(e)} \xi, \\ \nabla_e e &= \nabla_{\varphi(e)} \varphi(e) = \nabla_\xi \xi = 0, \end{aligned}$$

then the covariant derivative for M in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$ is defined by

$$\begin{aligned} \nabla_X Y &= X[y_1]\, e + X[y_2]\, \varphi(e) + X[y_3]\, \xi \\ &\quad -\eta(Y)\varphi(X) - \eta(X)\varphi(Y) - d\eta(X,Y)\xi, \end{aligned} \tag{3.10}$$

(see [4, 8]).

Proposition 2. *Let $M : X(u,v)$ be a surface in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$. The second order-derivatives X_{uu}, X_{uv} and X_{vv} are, respectively,*

$$\begin{aligned} X_{uu} &= \frac{1}{2}[f_{2,uu} + f_{1,u}(f_{3,u} - f_2 f_{1,u})]\, e + \frac{1}{2}[f_{1,uu} - f_{2,u}(f_{3,u} - f_2 f_{1,u})]\, \varphi(e) \\ &\quad + \frac{1}{2}[f_{3,uu} - f_{2,u} f_{1,u} - f_{1,uu} f_2]\, \xi, \end{aligned} \tag{3.11}$$

$$\begin{aligned} X_{uv} &= \frac{1}{2}\left[f_{2,uv} + \frac{1}{2} f_{1,v}(f_{3,u} - f_2 f_{1,u}) + \frac{1}{2} f_{1,u}(f_{3,v} - f_2 f_{1,v})\right] e \\ &\quad + \frac{1}{2}\left[f_{1,uv} - \frac{1}{2} f_{2,v}(f_{3,u} - f_2 f_{1,u}) - \frac{1}{2} f_{2,u}(f_{3,v} - f_2 f_{1,v})\right] \varphi(e) \\ &\quad + \frac{1}{2}\left[f_{3,uv} - \frac{1}{2} f_{2,v} f_{1,u} - f_{1,uv} f_2 - \frac{1}{2} f_{1,v} f_{2,u}\right] \xi, \end{aligned} \tag{3.12}$$

$$\begin{aligned} X_{vv} &= \frac{1}{2}\left[f_{2,vv} + f_{1,v}(f_{3,v} - f_2 f_{1,v})\right] e \\ &\quad + \frac{1}{2}\left[f_{1,vv} - f_{2,v}(f_{3,v} - f_2 f_{1,v})\right] \varphi(e) \\ &\quad + \frac{1}{2}\left[f_{3,vv} - f_{2,v} f_{1,v} - f_{1,vv} f_2\right] \xi. \end{aligned} \tag{3.13}$$

where $f_{i,uu}$, $f_{i,uv}$ and $f_{i,vv}$ $(1 \leq i \leq 3)$ mean that the second derivatives of $f_i(u,v)$ according to the u and $v-$parameters.

Proof. From the definition of covariant derivative

$$\begin{aligned} X_{uu} &= \nabla_{X_u} X_u \\ &= \frac{1}{2} f_{2,uu} e + \frac{1}{2} f_{1,uu} \varphi(e) + \frac{1}{2}(f_{3,uu} - f_{2,u} f_{1,u} - f_{1,uu} f_2)\xi \\ &\quad -\eta(X_u)\varphi(X_u) - \eta(X_u)\varphi(X_u) - g(X_u, \varphi(X_u))\xi \\ &= \frac{1}{2} f_{2,uu} e + \frac{1}{2} f_{1,uu} \varphi(e) + \frac{1}{2}(f_{3,uu} - f_{2,u} f_{1,u} - f_{1,uu} f_2)\xi \\ &\quad -2\eta(X_u)\varphi(X_u) - g(X_u, \varphi(X_u))\xi \\ &= \frac{1}{2}\left[f_{2,uu} + f_{1,u}(f_{3,u} - f_2 f_{1,u})\right] e + \frac{1}{2}\left[f_{1,uu} - f_{2,u}(f_{3,u} - f_2 f_{1,u})\right] \varphi(e) \\ &\quad + \frac{1}{2}\left[f_{3,uu} - f_{2,u} f_{1,u} - f_{1,uu} f_2\right] \xi \end{aligned}$$

and similarly we can easily see that

$$\begin{aligned} X_{uv} &= \frac{1}{2}\left[f_{2,uv} + \frac{1}{2} f_{1,v}(f_{3,u} - f_2 f_{1,u}) + \frac{1}{2} f_{1,u}(f_{3,v} - f_2 f_{1,v})\right] e \\ &\quad + \frac{1}{2}\left[f_{1,uv} - \frac{1}{2} f_{2,v}(f_{3,u} - f_2 f_{1,u}) - \frac{1}{2} f_{2,u}(f_{3,v} - f_2 f_{1,v})\right] \varphi(e) \\ &\quad + \frac{1}{2}\left[f_{3,uv} - \frac{1}{2} f_{2,v} f_{1,u} - f_{1,uv} f_2 - \frac{1}{2} f_{1,v} f_{2,u}\right] \xi, \end{aligned}$$

$$\begin{aligned} X_{vv} &= \frac{1}{2}\left[f_{2,vv} + f_{1,v}(f_{3,v} - f_2 f_{1,v})\right] e \\ &\quad + \frac{1}{2}\left[f_{1,vv} - f_{2,v}(f_{3,v} - f_2 f_{1,v})\right] \varphi(e) \\ &\quad + \frac{1}{2}\left[f_{3,vv} - f_{2,v} f_{1,v} - f_{1,vv} f_2\right] \xi. \end{aligned}$$

These complete the proof. □

Theorem 3.5. *Let $M : X(u,v)$ be a surface in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$. The shape operator matrix of M in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$ is*

$$S = \begin{bmatrix} \frac{Gl-Fm}{EG-F^2} & \frac{Em-Fl}{EG-F^2} \\ \frac{Gm-Fn}{EG-F^2} & \frac{En-Fm}{EG-F^2} \end{bmatrix} \tag{3.14}$$

where $l = g(N, X_{uu})$, $m = g(N, X_{uv})$ *and* $n = g(N, X_{vv})$.

Proof. We need expressions of $S(X_u)$ and $S(X_v)$ in terms of the basis for $\{X_u, X_v\}$. We can write $S(X_u) = aX_u + bX_v$ and $S(X_v) = cX_u + dX_v$. Our aim is to find a, b, c and d. If we can compute $g(S(X_u), X_u)$ and $g(S(X_u), X_v)$, we find

$$a = \frac{Gl - Fm}{EG - F^2} \quad , \quad b = \frac{Em - Fl}{EG - F^2}$$

and similarly if we can compute $g(S(X_v), X_u)$ and $g(S(X_v), X_v)$, we know

$$c = \frac{Gm - Fn}{EG - F^2} \quad , \quad d = \frac{En - Fm}{EG - F^2}.$$

Consequently, since we know that

$$\begin{aligned} S(X_u) &= \frac{Gl - Fm}{EG - F^2} X_u + \frac{Em - Fl}{EG - F^2} X_v \\ S(X_v) &= \frac{Gm - Fn}{EG - F^2} X_u + \frac{En - Fm}{EG - F^2} X_v \end{aligned}$$

we have

$$S = \begin{bmatrix} \frac{Gl-Fm}{EG-F^2} & \frac{Em-Fl}{EG-F^2} \\ \frac{Gm-Fn}{EG-F^2} & \frac{En-Fm}{EG-F^2} \end{bmatrix}$$

where the matrix is in terms of the basis for $\{X_u, X_v\}$. These complete the proof. □

Theorem 3.6. *Let* $M : X(u, v)$ *be a surface in* $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$. *According to the shape operator matrix of the surface, the Gaussian curvature of* M *is*

$$K = \frac{ln - m^2}{EG - F^2} \tag{3.15}$$

where l, n, m, E, G *and* F *are defined in the above equalities.*

Proof. From the definition of Gaussian curvature K for the matrix $S = \begin{bmatrix} \frac{Gl-Fm}{EG-F^2} & \frac{Em-Fl}{EG-F^2} \\ \frac{Gm-Fn}{EG-F^2} & \frac{En-Fm}{EG-F^2} \end{bmatrix}$, we may write that

$$\begin{aligned} K &= \det S, \\ &= \left(\frac{Gl-Fm}{EG-F^2}\right)\left(\frac{En-Fm}{EG-F^2}\right) - \left(\frac{Em-Fl}{EG-F^2}\right)\left(\frac{Gm-Fn}{EG-F^2}\right) \\ &= \frac{EGln - FGmn - EFmn + F^2m^2 - EGm^2 + EFmn + GFml - F^2ln}{(EG-F^2)^2} \\ &= \frac{EG\left(ln-m^2\right) - F^2\left(ln-m^2\right)}{(EG-F^2)^2} \\ &= \frac{\left(EG-F^2\right)\left(ln-m^2\right)}{(EG-F^2)^2} \\ &= \frac{ln-m^2}{EG-F^2} \end{aligned}$$

which completes the proof. □

Theorem 3.7. *Let $M : X(u,v)$ be a surface in $(\mathbb{R}^3(-3), \varphi, \xi, \eta, g)$. According to the shape operator matrix of the surface, the mean curvature of M is*

$$H = \frac{1}{2}\left(\frac{Gl+En-2Fm}{EG-F^2}\right) \tag{3.16}$$

where l, n, m, E, G and F are defined previously.

Proof. From The definition of mean curvature H for the matrix $S = \begin{bmatrix} \frac{Gl-Fm}{EG-F^2} & \frac{Em-Fl}{EG-F^2} \\ \frac{Gm-Fn}{EG-F^2} & \frac{En-Fm}{EG-F^2} \end{bmatrix}$, we may write that

$$\begin{aligned} H &= \frac{1}{2}\mathrm{tr}S = \frac{1}{2}\left(\frac{Gl-Fm+En-Fm}{EG-F^2}\right) \\ H &= \frac{1}{2}\left(\frac{Gl+En-2Fm}{EG-F^2}\right), \end{aligned}$$

which completes the proof. □

ÖZET: Bu makalede, 3-boyutlu hemen hemen kontak manifoldlarda Camcı [6] tarafından tanımlanan dış çarpım yardımıyla yüzeyler gözönünde bulunduruldu. Camcı, çalışmasında tanımladığı bu dış çarpımı kullanarak bu tip manifoldlarda eğriler teorisini çalıştı. Bu çalışmada $\mathbb{R}^3(-3)$ uzayında herhangi bir yüzeyin birim normal vektör alanı tanımlandı ve bu yüzeye ait şekil operatörü matrisi araştırıldı. Dahası, bu yüzeyin Gauss ve ortalama eğriliklerinin formülleri hesaplandı.

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