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NEW CRITERIA FOR MEROMORPHIC CONVEX FUNCTIONS

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ABSTRACT. Let $Q(\alpha)$ be the class of functions of the form

$$f(z)=\frac{a_{-1}}{z}+\sum_{k=0}^{\infty}a_kz^k \quad (a_{-1}\neq 0)$$

which are regular in the punctured disc $U^*=\{z:0<|z|<1\}$ and satisfying

$$\operatorname{Re}\left\{\frac{\left(D^{n+1}f(z)\right)'}{\left(D^nf(z)\right)'}-2\right\}<-\alpha,\quad 0\le\alpha<1,\ |z|<1,$$

and $n\in N_0=\{0,1,2,...\}$, where

$$D^nf(z)=\frac{a_{-1}}{z}+\sum_{k=2}^{\infty}k^na_{k-2}z^{k-2}.$$

It is proved that $Q_{n+1}(\alpha)\subset Q_n(\alpha)$. Since $Q_0(\alpha)$ is the class of meromorphically convex functions of order $\alpha(0\le\alpha<1)$ the members of $Q_n(\alpha)$ are meromorfically convex . Further property preserving integrals are considered.

1. INTRODUCTION

Let $\sum$ denotes the class of functions of the form

$$f(z)=\frac{a_{-1}}{z}+\sum_{k=0}^{\infty}a_kz^k \quad (a_{-1}\neq 0) \tag{1.1}$$

which are regular in the punctured disc $U^*=\{z:0<|z|<1\}$.
Define

$$D^0f(z)=f(z), \tag{1.2}$$

$$\begin{aligned}D^1f(z) &= \frac{a_{-1}}{z}+2a_o+3a_1z+4a_2z^2+...\\ &= \frac{\left(z^2f(z)\right)'}{z},\end{aligned} \tag{1.3}$$

$$D^2f(z)=D\left(D^1f(z)\right), \tag{1.4}$$

and for $n=1,2,3,...$

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$$D^n f(z) = D\left(D^{n-1} f(z)\right) = \frac{a_{-1}}{z} + \sum_{m=2}^{\infty} m^n a_{m-2} z^{m-2} = \frac{\left(z^2 D^{n-1} f(z)\right)'}{z}. \quad (1.5)$$

In this paper, we shall show that a function $f(z)$ in $\sum$, which satisfies are of the conditions

$$\mathrm{Re}\left\{\frac{\left(D^{n+1} f(z)\right)'}{\left(D^n f(z)\right)'} - 2\right\} < -\alpha, \quad |z| < 1, \quad (1.6)$$

$0 \leq \alpha < 1$, and $n \in N_0 = \{0, 1, 2, ...\}$, is convex of order α in $0 < |z| < 1$. More precisely it is proved that for the classes $Q_n(\alpha)$ of functions in $\sum$ satisfying (1.6),

$$Q_{n+1}(\alpha) \subset Q_n(\alpha) \quad (n \in N_0,\ 0 \leq \alpha < 1) \quad (1.7)$$

holds.Since $Q_0(\alpha)$ equals $\sum_k(\alpha)$ (the class of meromorphically convex functions of order α, $(0 \leq \alpha < 1)$ the convexity of members of $Q_n(\alpha)$ is a consequence of (1.7). Further property preserving integrals are considered, a known result of Goel and Sohi [2,Corollary 1] is obtained as a particular case and a result of Bajapi [1, Theorem 1] is extended.

In [4] Uralegaddi and Somanatha obtained a new criterion for meromorphic starlike univalent functions via the basic inclusion relationship $B_{n+1}(\alpha) \subset B_n(\alpha)$, $0 \leq \alpha < 1$ and $n \in N_0$, where $B_n(\alpha)$ is the class of function $f(z) \in \sum$ satisfying

$$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 2\right\} < -\alpha,$$

$0 \leq \alpha < 1$, $n \in N_0$ and $|z| < 1$.

2. Properties of the class $Q_n(\alpha)$

In proving our main results (Theorem 1 and Theorem 2 below), we shall need the following lemma due to Jack [3].

Lemma 2.1. *Let $w(z)$ be non-constant regular in $U = \{z : |z| < 1\}, w(0) = 0$. If $w(z)$ attains its maximum value on the circle $|z| = r < 1$ at z_o, we have $z_o w'(z_o) = k w(z_o)$, where k is a real number, $k \geq 1$.*

Theorem 2.2. *$Q_{n+1}(\alpha) \subset Q_n(\alpha)$ for each integer $n \in N_0$.*

Proof. Let $f(z) \in Q_{n+1}(\alpha)$ .Then

$$\mathrm{Re}\left\{\frac{\left(D^{n+2} f(z)\right)'}{\left(D^{n+1} f(z)\right)'} - 2\right\} < -\alpha, \quad |z| < 1. \quad (2.1)$$

We have to show that (2.1) implies the inequality

$$\mathrm{Re}\left\{\frac{\left(D^{n+1} f(z)\right)'}{\left(D^n f(z)\right)'} - 2\right\} < -\alpha. \quad (2.2)$$

Define a regular function $w(z)$ in the unit disc $U = \{z : |z| < 1\}$ by

$$\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'} - 2 = -\frac{1+(2\alpha-1)\,w(z)}{1+w(z)}. \tag{2.3}$$

Clearly $w(0) = 0$. Equation (2.3) may be written as

$$\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'} = \frac{1+(3-2\alpha)\,w(z)}{1+w(z)}. \tag{2.4}$$

Differentiating (2.4) logarithmically, we obtain

$$\frac{z\left(D^{n+1}f(z)\right)''}{\left(D^{n+1}f(z)\right)'} - \frac{z\left(D^{n}f(z)\right)''}{\left(D^{n}f(z)\right)'} = \frac{(3-2\alpha)\,zw'(z)}{1+(3-2\alpha)\,w(z)} - \frac{zw'(z)}{1+w(z)}. \tag{2.5}$$

From the following identity

$$z\left(D^{n}f(z)\right)' = D^{n+1}f(z) - 2D^{n}f(z) \tag{2.6}$$

we have

$$z\left(D^{n}f(z)\right)'' = \left(D^{n+1}f(z)\right)' - 3\left(D^{n}f(z)\right)'. \tag{2.7}$$

Using the identity (2.7), equation (2.5) may be written as

$$\frac{\frac{\left(D^{n+2}f(z)\right)'}{\left(D^{n+1}f(z)\right)'} - 2 + \alpha}{1-\alpha} = \frac{2zw'(z)}{(1+w(z))\,[1+(3-2\alpha)\,w(z)]} - \frac{1-w(z)}{1+w(z)} \tag{2.8}$$

We claim that $|w(z)| < 1$ for $z \in U$. For otherwise by the above lemma there exists $z_o, |z_o| < 1$ such that

$$z_o w'(z_o) = kw(z_o), \tag{2.9}$$

where $|w(z_o)| = 1$ and $k \geq 1$. From (2.8) and (2.9) we obtain

$$\frac{\frac{\left(D^{n+2}f(z_o)\right)'}{\left(D^{n+1}f(z_o)\right)'} - 2 + \alpha}{1-\alpha} = \frac{2kw(z_o)}{(1+w(z_o))\,[1+(3-2\alpha)\,w(z_o)]} - \frac{1-w(z_o)}{1+w(z_o)}. \tag{2.10}$$

Thus

$$\operatorname{Re}\left\{\frac{\frac{\left(D^{n+2}f(z_o)\right)'}{\left(D^{n+1}f(z_o)\right)'} - 2 + \alpha}{1-\alpha}\right\} \geq \frac{1}{2(1-\alpha)} > 0$$

which contradicts (2.1). Hence $|w(z)| < 1$ for $z \in U$ and from (2.2) it follows that $f(z) \in Q_n(\alpha)$. □

Theorem 2.3. *Let $f(z) \in \sum$ and for a given $n \in N_0$ and $c > 0$, satisfy the condition*

$$\operatorname{Re}\left\{\frac{\left(D^{n+1}f(z)\right)'}{\left(D^{n}f(z)\right)'} - 2\right\} < -\alpha + \frac{1-\alpha}{2(1-\alpha+c)}. \tag{2.11}$$

Then

$$F(z)=\frac{c}{z^{c+1}}\int_0^z t^c f(t)\,dt \tag{2.12}$$

belongs to $Q_n(\alpha)$.

Proof. From the definition of $F(z)$ we have

$$z\left(D^nF(z)\right)'=cD^nf(z)-(c+1)\,D^nF(z) \tag{2.13}$$

and also

$$z\left(D^nF(z)\right)'=D^{n+1}F(z)-2D^nF(z). \tag{2.14}$$

Using (2.13) and (2.14) the condition (2.11) may be written as

$$\mathrm{Re}\left\{\frac{\frac{\left(D^{n+2}F(z)\right)'}{\left(D^{n+1}F(z)\right)'}+(c-1)}{1+(c-1)\frac{D^nF(z)}{D^{n+1}F(z)}}-2\right\}<-\alpha+\frac{1-\alpha}{2(1-\alpha+c)}. \tag{2.15}$$

We have to prove that (2.14) implies the inequality

$$\mathrm{Re}\left\{\frac{\left(D^{n+1}F(z)\right)'}{\left(D^nF(z)\right)'}-2\right\}<-\alpha.$$

Define a regular function $w(z)$ in the unit disc U by

$$\frac{\left(D^{n+1}F(z)\right)'}{\left(D^nF(z)\right)'}-2=\frac{1+(2\alpha-1)\,w(z)}{1+w(z)}, \tag{2.16}$$

clearly $w(0)=0$. The equation (2.16) may be written as

$$\frac{\left(D^{n+1}F(z)\right)'}{\left(D^nF(z)\right)'}=\frac{1+(3-2\alpha)\,w(z)}{1+w(z)}. \tag{2.17}$$

Differentiating (2.17) logarithmically and simplifying we obtain

$$\frac{\frac{\left(D^{n+2}F(z)\right)'}{\left(D^{n+1}F(z)\right)'}+(c-1)}{1+(c-1)\frac{\left(D^nF(z)\right)'}{\left(D^{n+1}F(z)\right)'}}-2=-\left[\alpha+(1-\alpha)\frac{1-w(z)}{1+w(z)}\right]+\frac{2(1-\alpha)\,zw'(z)}{(1+w(z))\left[c+(2-2\alpha+c)\,w(z)\right]}. \tag{2.18}$$

The remaining part of the proof is similar to that of Theorem2. □

Puting $n=0$ and $c=1$ in Theorem 3, we obtain the following result:

Corollary 1. *If* $f(z)\in\sum$ *and satisfies*

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}<-\alpha+\frac{1-\alpha}{2(2-\alpha)}$$

then

$$F(z) = \frac{1}{z^2} \int_0^z t f(t)\, dt$$

belongs to $\sum_k(\alpha)$.

Putting $\alpha = 0$ and $a_{-1} = 1$ in Corollary 1 we obtain the following result of Goel and Sohi [2].

Corollary 2. *If* $f(z) = \frac{1}{z} + \sum\limits_{k=0}^{\infty} a_k z^k \in \sum$ *satisfies*

$$\mathrm{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} < \frac{1}{4}$$

then

$$F(z) = \frac{1}{z^2} \int_0^z t f(t)\, dt$$

belongs to $\sum_k$.

Remark 2.4. Corollary 2 extends a result of Bajpai [1].

Theorem 2.5. $f(z) \in Q_n(\alpha)$ *if and only if*

$$F(z) = \frac{1}{z^2} \int_0^z t f(t)\, dt \in Q_{n+1}(\alpha).$$

Proof. Proof. From the definition of $F(z)$, we have □

$$D^n(zF'(z)) + 2D^n F(z) = D^n f(z),$$

that is,

$$z(D^n F(z))' + D^n F(z) = D^n f(z) \tag{2.19}$$

By using the identity (2.6), (2.19) reduces to $D^n f(z) = D^{n+1} F(z)$. Hence $(D^{n+1} f(z))' = (D^{n+2} F(z))'$.Therefore

$$\frac{(D^{n+1} f(z))'}{(D^n f(z))'} = \frac{(D^{n+2} F(z))'}{(D^{n+1} F(z))'}$$

and the result follows. □

ÖZET: $Q(\alpha)$, *disc* $U^* = \{z : 0 < |z| < 1\}$ delik yuvarında regüler olan ve $n \in N_0 = \{0, 1, 2, \ldots\}$ için

$$D^n f(z) = \frac{a_{-1}}{z} + \sum_{k=2}^{\infty} k^n a_{k-2} z^{k-2}.$$

olmak üzere

$$\mathrm{Re}\left\{\frac{(D^{n+1} f(z))'}{(D^n f(z))'} - 2\right\} < -\alpha,\ 0 \le \alpha < 1, |z| < 1,$$

koşulunu gerçekleyen

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-1} \neq 0)$$

formundaki fonksiyonların sınıfı olsun.

Bu çalışmada $Q_{n+1}(\alpha) \subset Q_n(\alpha)$ olduğu gösterilmiştir. $Q_0(\alpha)$,α yıncı mertebeden $(0 \leq \alpha < 1)$ meromorfik konveks fonksiyonların sınıfı olduğundan $Q_n(\alpha)$ nın elemanları da meromorfik konvekstir.

Ayrıca özelik koruyan integraller de incelenmiştir.

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