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ON THE HADAMARD PRODUCTS OF GCD AND LCM MATRICES

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ABSTRACT

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The matrix (S) having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry is called the greatest common divisor (GCD) matrix on S. The matrix $[S]$ having the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry is called the least common multiple (LCM) matrix on S. In this paper we obtain some results related with Hadamard products of GCD and LCM matrices. The set S is factor-closed if it contains every divisor of each of its elements. It is well-known,that if S is factor-closed,then there exit the inverses of the GCD and LCM matrices on S. So we conjecture that if the set S is factor-closed, then $(S)o(S)^{-1}$ and $[S]o[S]^{-1}$ matrices are doubly stochastic matrices and $tr\left((S)o(S)^{-1}\right) = tr\left((S)\right) = \sum_{i=1}^{n} x_i$.

1. INTRODUCTION

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The matrix (S) having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j-entry is called the greatest common divisor (GCD) matrix on S. The study of GCD matrices was introduced by Beslin and Ligh [1]. They have shown that every GCD matrix is positive definite.

The matrix $[S]$ having the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j-entry is called the least common multiple (LCM) matrix on S. Smith [3] also considered the determinant of LCM matrix on a factor-closed set. We note that GCD matrix (S) and LCM matrix $[S]$ are invertible when S is factor-closed set. In this paper we obtained some results related with determinant, rank, norm and permanent of the Hadamard product of GCD matrix (S) and LCM matrix [S]. Moreover we conjecture that if S is factor-closed, then the Hadamard product of GCD matrix (S) and the inverse of GCD matrix (S) is a doubly stochastic matrix and also the Hadamard product of LCM matrix [S] and the inverse of LCM matrix [S] is a doubly stochastic matrix. Again we conjecture that if S is factor-closed, then $\mathrm{tr}\left((S)o(S)^{-1}\right) = \mathrm{tr}\left((S)\right) = \sum_{i=1}^{n} x_i$.

2. MAIN RESULTS

Definition 1. The Hadamard product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size is just their element-wise product $AoB = (a_{ij}b_{ij})$.

Lemma 1. Let $S = \{x_1, x_2,\ldots, x_n\}$ be a set of distinct positive integers.If (S) is the GCD matrix and [S] is the LCM matrix defined on S, then

$$\left((S)o[S]\right)_{ij} = \begin{cases} x_i^2 & \text{if } i = j \\ x_i x_j & \text{if } i \neq j \end{cases}.$$

Proof. Consider the set S and if we denote the greatest common divisor of x_i and x_j with (x_i, x_j) and the least common multiple of x_i and x_j with $[x_i, x_j]$, then we have

$$(x_i, x_j)[x_i, x_j] = x_i x_j, \ (i, j = 1,2,\ldots,n.).$$

Therefore by the definition 1, we get the proof.

Remark 1. Clearly (S)o[S] matrix is symmetric.

Theorem 1. Let $S = \{x_1, x_2,\ldots, x_n\}$ be a set of distinct positive integers. If (S) is the GCD matrix and [S] is the LCM matrix defined on S, then

$$\det((S)o[S]) = 0.$$

Proof. By Lemma 1 and using the properties of determinants we have

$$\det((S)o[S]) = \det\begin{bmatrix} x_1^2 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & x_2^2 & \dots & x_2x_n \\ \dots & \dots & \dots & \dots \\ x_nx_1 & x_nx_2 & \dots & x_n^2 \end{bmatrix}$$

$$= (x_1x_2\dots x_n)\det\begin{bmatrix} x_1 & x_2 & \dots & x_n \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \end{bmatrix}$$

$$= (x_1x_2\dots x_n)\cdot 0$$

$$= 0$$

and thus the proof is complete.

Definition 2. Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. The matrix $1/(S)$ is the $n\times n$ matrix whose i, j-entry is $\dfrac{1}{(x_i, x_j)}$. We call $1/(S)$ the reciprocal GCD matrix on S.

Corollary 1. Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. If (S) is the GCD matrix and $1/(S)$ is the reciprocal GCD matrix defined on S, then

$$\det((S)o1/(S)) = 0.$$

Proof. Since $(S)o1/(S)$ is the matrix whose all entries are equal to 1, it is easily seen that

$$\det((S)o1/(S)) = 0.$$

Theorem 2. If (S) and [S] are the GCD and LCM matrices defined on $S = \{x_1, x_2, \ldots, x_n\}$ the set of distinct positive integers, respectively, then

$$\operatorname{rank}((S)o[S]) = 1.$$

Proof. Firstly by Theorem 1 we say that $\operatorname{rank}((S)o[S]) < n$. If we denote the rows of $(S)o[S]$ with r_i $(i=1,2,\ldots n)$, then we can write

$$\begin{aligned} r_1 &= x_1(x_1, x_2, \ldots, x_n) \\ r_2 &= x_2(x_1, x_2, \ldots, x_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ r_n &= x_n(x_1, x_2, \ldots, x_n). \end{aligned}$$

So we write

$$r_1 = \frac{x_1}{x_{i+1}} r_{i+1} \qquad i = 1,2,\ldots,n-1.$$

Therefore r_1 is a multiple of the rows $r_2, r_3, \ldots, r_n$. Hence by the elementary row operations it follows that the number of row which isn't zero of the matrix $(S)o[S]$ is 1 and thus the proof is complete.

Definition 3. Let $A = (a_{ij})$ be $n \times n$ matrix over any commutative ring. The permanent of A written per(A), or simply PerA, is defined by

$$\operatorname{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all one-to-one functions from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,n\}$.

Now we can state the following result.

Theorem 3. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers. If (S) and [S] are the GCD and LCM matrices defined on S, respectively, then

$$\operatorname{per}((S)o[S]) = n!\prod_{i=1}^{n} x_i^2 .$$

Proof. For the brevity if we take C=(S)o[S] and $C = (c_{ij})$ then by the Definition 3, we have

$$\operatorname{per}(C) = \sum_{\sigma} c_{1\sigma(1)} c_{2\sigma(2)} \ldots c_{n\sigma(n)} .$$

On the other hand we note that the sequence $(c_{1\sigma(1)}, c_{2\sigma(2)}, \ldots, c_{n\sigma(n)})$ is called a diagonal of C, and the product $c_{1\sigma(1)} c_{2\sigma(2)} \ldots c_{n\sigma(n)}$ is a diagonal product of C. Thus the permanent of C is the sum of all diagonal products of C. Considering the structure of the matrix $C = (S)o[S]$ we get result.

Corollary 2. If (S) is the GCD matrix defined on the set S of distinct positive integers, and $1/(S)$ is the reciprocal matrix of GCD matrix, then

$$\operatorname{per}((S)o1/(S)) = n!$$

Proof. Since $(S)o1/(S)$ is the matrix whose all entries are equal to 1, it is easily seen that

$$\operatorname{per}((S)o1/(S)) = n!$$

Definition 4.(i) The $_1$ norm is defined for $A \in M_n$ by

$$\|A\|_1 = \sum_{i,j=1}^{n} |a_{ij}| .$$

(ii) The Euclidean norm or $_2$ norm is defined for $A \in M_n$ by

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

(iii) The ∞ norm is defined for $A \in M_n$ by

$$\|A\|_\infty = \max_{1 \le i,j \le n} |a_{ij}|.$$

(iv) The maximum row sum matrix norm is defined $A \in M_n$ by

$$\|A\|_r = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

(v) The maximum column sum matrix norm is defined for $A \in M_n$ by

$$\|A\|_c = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$

Theorem 4. Let $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of distinct positive integers. If (S) and $[S]$ are the GCD and LCM matrices defined on S, respectively, then the following statements are satisfied:

(i) $\| (S)o[S] \|_1 = \left(\sum_{i=1}^{n} x_i \right)^2$

(ii) $\| (S)o[S] \|_2 = \sum_{i=1}^{n} x_i^2$

(iii) $\| (S)o[S] \|_\infty = x_n^2$

(iv) $\| (S)o[S] \|_r = \| (S)o[S] \|_c = x_n \left(\sum_{i=1}^{n} x_i \right).$

Proof. If we denote i-th row sum of (S)o[S] with r_{S_i} (I=1,2,…,n). Then we can write

$$r_{S_1} = x_1\left(\sum_{i=1}^{n} x_i\right)$$

$$r_{S_2} = x_2\left(\sum_{i=1}^{n} x_i\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$r_{S_n} = x_n\left(\sum_{i=1}^{n} x_i\right).$$

So we have

$$\begin{aligned}\|(S)o[S]\|_1 &= r_{S_1} + r_{S_2} + \ldots + r_{S_n} \\ &= x_1\left(\sum_{i=1}^{n} x_i\right) + x_2\left(\sum_{i=1}^{n} x_i\right) + \ldots + x_n\left(\sum_{i=1}^{n} x_i\right) \\ &= (x_1 + x_2 + \ldots + x_n)\left(\sum_{i=1}^{n} x_i\right) \\ &= \left(\sum_{i=1}^{n} x_i\right)^2.\end{aligned}$$

(ii) Again for the brevity let take as C = (S)o[S] and $C = (c_{ij})$. Then we have

$$\mathrm{tr}(CC^T) = \sum_{i,j=1}^{n} |c_{ij}|^2 = \|C\|_2^2.$$

On the other hand from the structure of the matrix (S)o[S], it is easily seen that

$$\operatorname{tr}(CC^T) = \left(\sum_{i,j=1}^{n} x_i^{\,2}\right)^2.$$

So we obtain

$$\|C\|_2 = \sum_{i=1}^{n} x_i^{\,2}.$$

(iii) Since $S = \{x_1, x_2,\ldots, x_n\}$ is an ordered set of distinct positive integers, without loosing the generality we can assume that $x_1 < x_2 < \ldots < x_n$. Considering the definition of $_\infty$ norm we find

$$\|(S)o[S]\|_\infty = x_n^2.$$

(iv) Clearly since $(S)o[S]$ is symmetric, we have

$$\|(S)o[S]\|_r = \|(S)o[S]\|_c.$$

On the other hand we can assume that $x_1 < x_2 < \ldots < x_n$. Therefore by the definition maximum row sum matrix norm (or maximum column sum matrix norm) we have

$$\|(S)o[S]\|_r = \|(S)o[S]\|_c = x_n\left(\sum_{i=1}^{n} x_i\right).$$

Thus the Theorem 4 is completely proved.

Definition 5. A set $S = \{x_1, x_2,\ldots, x_n\}$ of positive integers is said to be factor-closed (FC) is whenever x_i is in S and d divides x_i, then d is in S.

The above definition is due to J.J. Malone.

Remark 2. We note that if the set $S = \{x_1, x_2, \ldots, x_n\}$ of distinct positive integers is not factor-closed, then an LCM matrix may not be invertible. But the GCD matrix (S) defined on any set S of distinct positive integers is always invertible.

Theorem 5. [2] Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. If S is factor-closed, then the inverse of the GCD matrix (S) defined on S is the matrix $(S)^{-1} = (t_{ij})$, where

$$t_{ij} = \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{\varphi(x_k)} \mu(x_k / x_i)\mu(x_k / x_j),$$

φ (.) is Euler's totient function and μ (.) denotes Moebius function.

Now we present the following.

Conjecture 1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $(S)^{-1}$ be the inverse of the GCD matrix (S) defined on S. If S is factor-closed, then $(S)o(S)^{-1}$ is a doubly stochastic matrix, i.e.,

$$\sum_{j=1}^{n} \left\{ (x_i, x_j) \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{\varphi(x_k)} \mu(x_k / x_i)\mu(x_k / x_j) \right\} = 1 \qquad (i = 1,2,\ldots,n)$$

and

$$\sum_{i=1}^{n} \left\{ (x_i, x_j) \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{\varphi(x_k)} \mu(x_k / x_i)\mu(x_k / x_j) \right\} = 1 \qquad (j = 1,2,\ldots,n).$$

Theorem 6. [2] Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. If S is factor-closed, then the inverse of the LCM matrix [S] defined on S is the matrix , where

$$b_{ij} = \frac{1}{x_i x_j} \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{g(x_k)} \mu(x_k / x_i) \mu(x_k / x_j),$$

and g is defined for each positive integer m by

$$g(m) = \frac{1}{m} \sum_{d|m} d\mu(d).$$

Conjecture 2. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $[S]^{-1}$ be the inverse of the LCM matrix [S] defined on S. If S is factor-closed, then $[S]o[S]^{-1}$ is a doubly stochastic matrix, i.e.,

$$\sum_{j=1}^{n} \left\{ \frac{[x_i, x_j]}{x_i x_j} \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{g(x_k)} \mu(x_k / x_i) \mu(x_k / x_j) \right\} = 1 \qquad (i = 1, 2, \ldots, n)$$

and

$$\sum_{i=1}^{n} \left\{ \frac{[x_i, x_j]}{x_i x_j} \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{1}{g(x_k)} \mu(x_k / x_i) \mu(x_k / x_j) \right\} = 1 \qquad (j = 1, 2, \ldots, n).$$

Conjecture 3. . Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $(S)^{-1}$ be the inverse of the GCD matrix (S) defined on S. If S is factor-closed, then

$$tr\left((S)o(S)^{-1}\right) = tr((S)) = \sum_{i=1}^{n} x_i,$$

i.e.,

$$\sum_{i=1}^{n}\left\{x_i \sum_{x_i | x_k} \frac{1}{\varphi(x_k)} \mu^2(x_k / x_i)\right\} = \sum_{i=1}^{n} x_i .$$

Remark 3. The Conjecture 3 is not true for $[S]o[S]^{-1}$ matrix, i.e.,

$$\mathrm{tr}\left([S]o[S]^{-1}\right) \neq \mathrm{tr}([S]) = \sum_{i=1}^{n} x_i .$$

For example, if S = {1, 2, 4} (we note that the set S is factor-closed), then we have the following:

$$[S] = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 4 \\ 4 & 4 & 4 \end{bmatrix}, \ [S]^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & \frac{-3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{-1}{4} \end{bmatrix}$$

and

$$[S]o[S]^{-1} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}.$$

Thus since $\mathrm{tr}\left([S]o[S]^{-1}\right) = -5$ and $\mathrm{tr}\left([S]\right) = \sum_{i=1}^{n} x_i = 7$, it follows that

$$\mathrm{tr}\left([S]o[S]^{-1}\right) \neq \mathrm{tr}([S]).$$

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