

SOME APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS TO A NEW CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

The object of the present paper is to prove various distortion theorems for the fractional calculus of the functions in the class $P(n, \lambda, \alpha, r)$ consisting of analytic functions with negative coefficients in the unit disk.

1. INTRODUCTION

Let $A(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} \alpha_k z^k \quad (\alpha_k \geq 0; n \in \mathbb{N} = \{1, 2, \dots\})$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. A function $f(z) \in A(n)$ is said to be in the class $P(n, \lambda, \alpha, r)$ if it satisfies

$$\operatorname{Re} \left\{ z \frac{(\lambda r z^{r-1} + 1 - \lambda) f'(z) + \lambda z^r f''(z)}{\lambda z^r f'(z) + (1 - \lambda) f(z)} \right\} > \alpha, \quad (r = 1, 2, \dots)$$

For some α ($0 \leq \alpha < 1$), λ ($0 \leq \lambda \leq 1$) and for all $z \in U$. Some properties of the class $P(n, \lambda, \alpha, r)$ were investigated by Kamali and Kadioglu [3].

Lemma 1. If a function $f(z) \in A(n)$ is in the class $P(n, \lambda, \alpha, r)$ then

$$\sum_{k=n+1}^{\infty} [(k - \alpha)(\lambda k - \lambda + 1) + \lambda k(r - 1)] a_k \leq (1 - \alpha) + \lambda(r - 1). \quad (1)$$

Proof. Let $f(z) \in P(n, \lambda, \alpha, r)$. We can write

$$\operatorname{Re} \left\{ z \frac{(\lambda r z^{r-1} + 1 - \lambda) f'(z) + \lambda z^r f''(z)}{\lambda z^r f'(z) + (1 - \lambda) f(z)} \right\} > \alpha.$$

Then

$$\begin{aligned} & \frac{[\lambda r z^{r-1} + (1 - \lambda)] [z - \sum_{k=n+1}^{\infty} k a_k z^k] + \lambda z^{r+1} [-\sum_{k=n+1}^{\infty} k(k-1) a_k z^{k-2}]}{\lambda z^r [1 - \sum_{k=n+1}^{\infty} k a_k z^{k-1}] + (1 - \lambda) [z - \sum_{k=n+1}^{\infty} a_k z^k]} \\ &= \frac{\lambda r z^r + (1 - \lambda) z - \sum_{k=n+1}^{\infty} [\lambda r k + \lambda k^2 - \lambda k] a_k z^{k+r-1} - \sum_{k=n+1}^{\infty} (1 - \lambda) k a_k z^k}{\lambda z^r + (1 - \lambda) z - \sum_{k=n+1}^{\infty} [\lambda k a_k z^{k+r-1} - \sum_{k=n+1}^{\infty} (1 - \lambda) a_k z^k]} \end{aligned}$$

If we choose z real and let $z \rightarrow 1^-$, we get

$$\sum_{k=n+1}^{\infty} [(k - \alpha)(\lambda k - \lambda + 1) + \lambda k(r - 1)] a_k \leq (1 - \alpha) + \lambda(r - 1).$$

2 Fractional Calculus

I begin with the statements of the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals) which were defined by Owa ([4], [5]) and were used recently by Srivastava and Owa, Altıntaş, and Cho and Auof ([1], [2], [6]).

Definition 2. The fractional integral of order δ is defined, for a function $f(z)$, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\delta}} d\zeta \quad (\delta > 0),$$

where $f(z)$ is an analytic function in a simply connected region of the z -plane containing the origin, and the multiplicity of $(z - \zeta)^{\delta-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Definition 3. The fractional derivative of order δ is defined, for a function $f(z)$, by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1 - \delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{\delta}} d\zeta \quad (0 \leq \delta < 1),$$

where $f(z)$ is constrained, and the multiplicity of $(z - \zeta)^{-\delta}$ is removed, as in Definition 2.

Theorem 4. Let the function $f(z)$ be in the class $P(n, \lambda, \alpha, r)$. Then we have

$$\left| D_z^{-\delta} f(z) \right| \geq \frac{|z|^{\delta}}{\Gamma(2 + \delta)} \left\{ |z| - \frac{[(1 - \alpha) + \lambda(r - 1)] \Gamma(n + 2) \Gamma(2 + \delta)}{[(n + 1 - \alpha)(\lambda n + 1) + \lambda(n + 1)(r - 1)] \Gamma(n + 2 + \delta)} |z|^{n+1} \right\}$$

and

$$|D_z^{-\delta} f(z)| \leq \frac{|z|^\delta}{\Gamma(2+\delta)} \left\{ |z| + \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2+\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2+\delta)} |z|^{n+1} \right\}$$

for $\delta > 0$ and $z \in U$. The result is sharp.

Proof. It is easy to see that

$$\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z) = z - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+\delta+1)} a_k z^k = z - \sum_{k=n+1}^{\infty} \psi(k) a_k z^k$$

where

$$\psi(k) = \frac{\Gamma(k+1)\Gamma(2+\delta)}{\Gamma(k+\delta+1)} \quad (k \geq n+1).$$

Noting that $\psi(k)$ is a decreasing function of k , we have

$$0 < \psi(k) \leq \psi(n+1) = \frac{\Gamma(n+2)\Gamma(2+\delta)}{\Gamma(n+2+\delta)}.$$

Using Theorem 1, we have

$$[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)] \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} [(k-\alpha)(\lambda k - \lambda + 1) + \lambda k(r-1)] a_k \leq (1-\alpha) + \lambda(r-1),$$

or

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{(1-\alpha) + \lambda(r-1)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]}.$$

We can see that

$$\begin{aligned} |\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| &\geq |z| - \psi(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\geq |z| - \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2+\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2+\delta)} |z|^{n+1} \end{aligned}$$

and

$$\begin{aligned} |\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| &\leq |z| + \psi(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\leq |z| + \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2+\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2+\delta)} |z|^{n+1} \end{aligned}$$

which prove the inequalities of the theorem. Further, equalities are attained for the function $f(z)$ defined by

$$D_z^{-\delta}f(z) = \frac{z^\delta}{\Gamma(2+\delta)} \left\{ z - \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2+\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2+\delta)} z^{n+1} \right\}$$

or

$$f(z) = z - \frac{[(1-\alpha) + \lambda(r-1)]}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]} z^{n+1}.$$

Theorem 5. Let the function $f(z)$ be in the class $P(n, \lambda, \alpha, r)$. Then we have

$$|D_z^\delta f(z)| \geq \frac{|z|^{-\delta}}{\Gamma(2-\delta)} \left\{ |z| - \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2-\delta)} |z|^{n+1} \right\}$$

and

$$|D_z^\delta f(z)| \leq \frac{|z|^{-\delta}}{\Gamma(2-\delta)} \left\{ |z| + \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2-\delta)} |z|^{n+1} \right\}$$

for $0 \leq \delta < 1$ and $z \in U$. The result is sharp.

Proof. It is easy to see that

$$\Gamma(2-\delta)z^\delta D_z^\delta f(z) = z - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k-\delta+1)} a_k z^k = z - \sum_{k=n+1}^{\infty} \psi(k) k a_k z^k$$

where

$$\psi(k) = \frac{\Gamma(k)\Gamma(2-\delta)}{\Gamma(k-\delta+1)} \quad (k \geq n+1).$$

Noting that $\psi(k)$ is a decreasing function of k , we have

$$0 < \psi(k) \leq \psi(n+1) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+2-\delta)}.$$

Using Theorem 1, we have

$$\frac{(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)}{n+1} \sum_{k=n+1}^{\infty} k a_k \leq (1-\alpha) + \lambda(r-1)$$

or

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)[(1-\alpha) + \lambda(r-1)]}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]}.$$

We can see that

$$\begin{aligned} |\Gamma(2-\delta)z^\delta D_z^\delta f(z)| &\geq |z| - \psi(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} k a_k \\ &\geq |z| - \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2-\delta)} |z|^{n+1} \end{aligned}$$

and

$$\begin{aligned} |\Gamma(2-\delta)z^\delta D_z^\delta f(z)| &\leq |z| + \psi(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} k a_k \\ &\leq |z| + \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2-\delta)} |z|^{n+1} \end{aligned}$$

which prove the inequalities of the theorem. Further, equalities are attained for the function $f(z)$ defined by

$$D_z^\delta f(z) = \frac{z^{-\delta}}{\Gamma(2-\delta)} \left\{ z - \frac{[(1-\alpha) + \lambda(r-1)]\Gamma(n+2)\Gamma(2-\delta)}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]\Gamma(n+2-\delta)} z^{n+1} \right\}$$

or

$$f(z) = z - \frac{[(1-\alpha) + \lambda(r-1)]}{[(n+1-\alpha)(\lambda n+1) + \lambda(n+1)(r-1)]} z^{n+1}.$$

Remark: If we take $r=1$ in this paper, then we have the result given by Altıntaş [1]

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