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UNIFORM CONVEXITY AND FOCAL POINTS

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ABSTRACT

In this paper the uniform convexity as well as strict uniform convexity have been defined for Riemannian manifolds. Necessary and sufficient conditions are established for a Riemannian manifold without conjugate points to be free from focal points in terms of the uniform convexity concept.

The convexity of spaces as well as subsets of spaces is one of the interesting area of research in both analysis and geometry [1, 2, 4, 6, 7].

The clasical concept of convexity of subsets may be given in the following. A subset $B \subset M$ in a Riemannian manifold M is convex if for each pair of points $p, q \in B$, there is a unique minimal geodesic segment [pq] from p to q and this segment is in B [3]. The subset $B \subset M$ is called strictly convex if it is convex and the boundary ∂B of B does not contain any geodesic segment [3]. There are other types of convexity for subsets in Riemannian manifold M [1].

Convexity of a space is mostly given in terms of convexity of its own balls as it will be considered below.

In (1979), Sastry and Naidu [7] defined and studied uniform convexity of metric linear spaces. A metric linear space E is uniformly convex if for any $\varepsilon>0, r>0$, there exists $\delta>0$ such that $d(x,0) \leq r+\delta$, $d(y,0) \leq r+\delta$ and $d(x,y) \geq \varepsilon$ imply $d(\frac{x+y}{2}, 0) \leq r$ [7]. Here d denotes the distance function in E.

In the present work we try to extend this definition of uniform convexity to any C^∞ Riemannian manifold M as given below. As a matter of fact our definition of uniform convexity is more suitable than that of

Sastry and Naidu [7] from the geometric point of view. The geometric conception enables us to derive some properties of Riemannian manifolds as we shall see later.

Definition 1. Let M be a complete C^∞ Riemannian manifold. M is said to be uniformly convex at the point $p \in M$ if for each pair of points $x,y \in M$, the midpoint z of any geodesic segment from x to y satisfies $d(p,z) \leq \max\{d(p,x),d(p,y)\}$. If $d(p,z)<\max\{d(p,x),d(p,y)\}$, then M is called strictly uniformly convex at p. If M is uniformly (resp. strictly uniformly) convex at all of its points then M is called uniformly (resp. strictly uniformly) convex.

Adopting our definition we can easily show that Euclidean space E^n $(n \geq 1)$ is strictly uniformly convex. On the contrary, the unit sphere $S^n \subset E^{n+1}$ $(n \geq 1)$ as well as the n-cylinder $S^{n-1} \times E^1 \subset E^{n+1}$ (n≥2] are not uniformly convex manifolds.

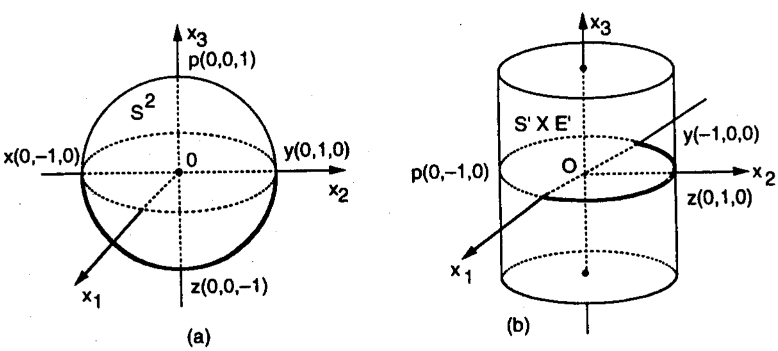

Fig. 1

From Fig. 1 we see that for the indicated points p,x,y and z

$$d(p,z) = \pi \quad , \quad d(p,x) = d(p,y) = \frac{\pi}{2}$$

therefore

$$d(p,z) > \max\{d(p,x),d(p,y)\}$$

which violates the condition of uniform convexity.

In a general Riemannian manifold M there may exist more than one geodesic segment between a pair of points $x,y\in M$. Hence to discuss the concept of uniform convexity in M it is necessary to consider all geodesic segments from x to y. Clearly, this would be a difficult matter in spite of being interesting. Therefore we consider, throughout this work, the same concept of convexity in a complete simply connected C^∞ Riemannian manifold W without conjugate points as for $x,y\in W$ there exists a unique geodesic segment [xy] joining this pair of points and leave considering M for future work. For the main geometric properties of W we refer the reader to [4, 5].

In a complete simply connected C^∞ Riemannian manifold W without conjugate points closed geodesic balls are not necessarily convex. The following result gives the sufficient condition of convexity of balls in W.

Proposition 2. Let W be a complete simply connected C^∞ Riemannian manifold without conjugate points. If W is uniformly convex, then each closed geodesic ball is a convex subset of W.

Proof. Let $\overline{B(a,\varepsilon)}\subset W$ be an arbitrary closed geodesic ball centered at $a\in W$ with radius $\varepsilon>0$. Assume on the contrary that $\overline{B(a,\varepsilon)}$ is not convex. Consequently, there exists a pair of points $p,q\in \overline{B(a,\varepsilon)}$ such that the closed geodesic segment $[pq]\subset\overline{B(a,\varepsilon)}$. Let us take [pq] to be the closed geodesic segment $\gamma:[0,1]\to W$ such that $p=\gamma(0)$ and $q=\gamma(1)$. As $\gamma([0,1])$ is not contained completely in

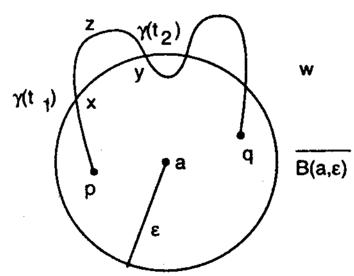

Fig. 2

$B(a,\varepsilon)$ and intersects $\partial B(a,\varepsilon)$ at least twice, then there exist $t_1,t_2\in[0,1]$ such that $t_1<t_2$, $x=\gamma(t_1)$, $y=\gamma(t_2)\in\partial B(a,\varepsilon)$ and $\gamma((t_1,t_2))\subset W\backslash B(a,\varepsilon)$ (see Fig. 2). Let z be the midpoint of the geodesic segment $\gamma([t_1,t_2])$. Clearly, $d(z,a)>\varepsilon$ while $d(a,\gamma(t_1))=d(a,\gamma(t_1))=\varepsilon$ which contradicts the assumption that W is a uniformly convex manifold.

Assuming W is uniformly convex at some point $p\in W$ we can prove, in the light of the above theorem, the following result.

Proposition 3. Let W be a complete simply connected C^∞ Riemannian manifold without conjugate points. If W is uniformly convex at some point $p \in W$, then every closed geodesic ball centered at p is a convex subset of W.

Now we can define another type of convex manifolds as follows: A Riemannian manifold M is called ball convex manifold if each closed geodesic ball in M is convex.

In the light of Proposition 2 we have

Corollary 4. Let W be a complete simply connected C^∞ Riemannian manifold without conjugate points. If W is uniformly convex, then W is a ball convex manifold.

Theorem 5. Let W be a strictly uniformly convex complete simply connected C^∞ Riemannian manifold without conjugate points. Then each closed geodesic ball in W is a strictly convex subset of W.

Proof. Using Proposition 2 and taking into account that every strictly uniformly convex manifold is uniformly convex we have that $\overline{B(a,\varepsilon)}$ is convex where $a \in W$ and $\varepsilon > 0$ are arbitrary. It remains now to show that $\overline{B(a,\varepsilon)}$ does not contain any geodesic segment on its boundary $\partial\overline{B(a,\varepsilon)}$. Assume on contrary that for $p,q \in \partial\overline{B(a,\varepsilon)}$, $[pq] \subset \partial\overline{B(a,\varepsilon)}$. Clearly, the mid-point z of [pq] belongs to $\partial\overline{B(a,\varepsilon)}$ and so $d(a,z) = d(a,p) = d(a,q)$. i.e. $d(a,z) \nless \max\{d(a,p), d(a,q)\}$ which contradicts the assumption that W is strictly uniformly convex.

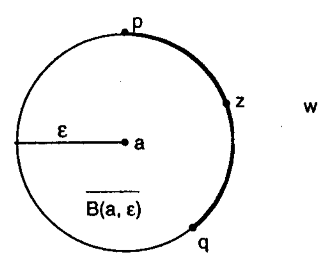

Fig. 3

The following result together with Theorem 7 below represent the main results of this work as they give necessary and sufficient conditions for a manifold without conjugate points to have no focal points.

Theorem 6. Let W be a strictly uniformly convex complete simply connected C^∞ Riemannian manifold without conjugate points, then W has no focal point.

Proof. Assume that W satisfies the assumptions given in the theorem. Firstly we show that any closed geodesic ball $\overline{B(a,\varepsilon)} \subset W$ has the following property. Let $p \in \partial B(a,\varepsilon)$ and $\gamma: [0,\infty] \to W$ a geodesic ray starting tangentially to $\partial B(a,\varepsilon)$ at $p=\gamma(0)$, then for each $t \in (0,\infty)$ we have $\gamma(t) \in W \backslash B(a,\varepsilon)$.

Assume on contrary that there exists $t_0 \in (0,\infty)$ such that $\gamma(t_0) \notin W \backslash B(a,\varepsilon)$. Therefore, we have the following possibilities represented in Fig. 4.

Notice that any other possibility may be reduced to one of these cases.

In the cases given by Fig. 4-a and Fig. 4-b we have that $\overline{B(a,\varepsilon)}$ is non-convex contradicting Proposition (2). Considering Fig. 4-c we obtain a contradiction to Theorem 5. For Fig. 4-d draw a thin geodesic cone C as indicated. We obtain a geodesic ray μ which is one of the generators of C starting from p such that $x \in \overline{B(a,\varepsilon)}$, $x \in \mu$ and $[px] \not\subset \overline{B(a,\varepsilon)}$ contradicting the convexity of $\overline{B(a,\varepsilon)}$ (Proposition 2).

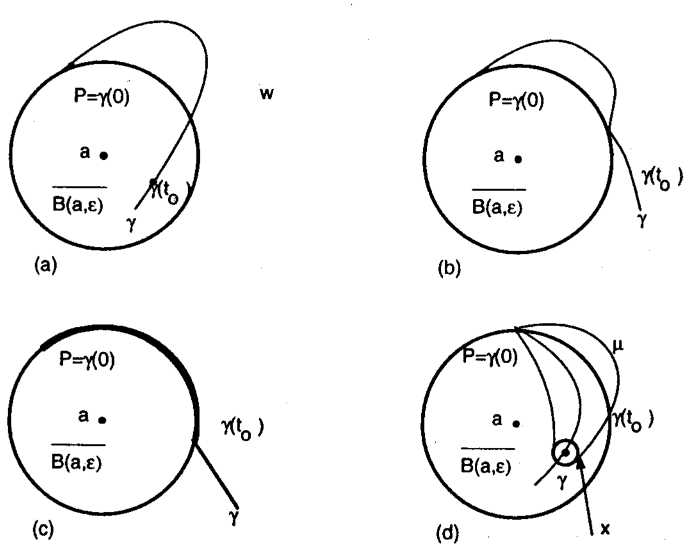

Fig. 4

In the light of the above discussion we have that if γ is a maximal geodesic in W which meets $\partial\overline{B}(a,\varepsilon)$ tangentially at some $p\in\partial\overline{B}(a,\varepsilon)$, then $\gamma\cap\overline{B}(a,\varepsilon)=\{p\}$.

Let γ, μ be maximal geodesics in W intersecting orthogonally at p. Take m to be an arbitrary point of μ and $m\in\gamma$. Draw the closed geodesic ball $\overline{B}(m,\lambda)$ where $\lambda=d(m,p)$. In the light of the above discussion we have that $\overline{B}(m,p)\cap\gamma=\{p\}$. Let us define the real function $d_m:\gamma\to R$ where $d_m(x)=d(m,x)$ is the distance from m to x. It is clear that $p\in\gamma$ is a global minimum point of d_m and hence γ has no focal points along μ. As μ is an arbitrary maximal geodesic perpendicular to γ, then γ has no focal points. Since γ is arbitrary we have that W is free from the focal points and the proof is complete.

For the geometry of manifolds without focal points see [5].

Theorem 7. Every complete simply connected C^∞ Riemannian manifold $\tilde{W}$ without focal points is strictly uniformly convex.

Proof. The proof depends basically on a result proved in [2] which states that in a complete simply connected C^∞ Riemannian manifold $\tilde{W}$ without focal points every geodesic ball is a strictly convex subset.

Assume on contrary to the theorem that $\tilde{W}$ is not strictly uniformly convex. Then there exists a point $p\in\tilde{W}$ and a pair of points $x,y\in\tilde{W}$ such that the midpoint z of [xy] satisfies $d(p,z)\geq\max\{d(p,x),d(p,y)\}$.

Without loss of generality assume firstly that $\max\{d(p,x),d(p,y)\} = d(p,x) = \lambda$. Draw a closed geodesic ball $\overline{B}(p,\lambda)$ which contains x and z on its boundary. We discuss the following possibilities:

(a) $d(p,z) = \lambda$

In this case $z\in\partial\overline{B}(p,\lambda)$. Draw a thin geodesic cone C with vertex x, axis [xy] and small base A about y where A is included in

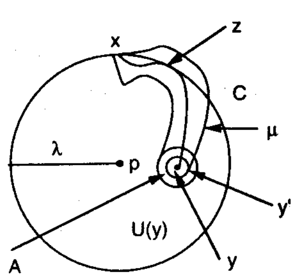

Fig. 5

some open neighborhood $U(y) \subset \overline{B(p,\lambda)}$ of y (see Fig. 5). From Fig. 5, it becomes clear that there exists a geodesic segment $\mu \subset C$ from x to $y' \subset U(y)$ such that μ is not included wholly in $\overline{B(p,\lambda)}$ which contradicts the convexity of $\overline{B(p,\lambda)}$. Therefore,

$$d(p,z) \neq \lambda.$$

(b) $d(p,z) > \lambda$

In this case $z \notin \overline{B(p,\lambda)}$ and consequently the geodesic segment [xy] which contains z will not be contained in $\overline{B(p,\lambda)}$. Therefore $\overline{B(p,\lambda)}$ is non-convex which is a contradiction. This contradiction shows that

$$d(p,z) \ngtr \lambda.$$

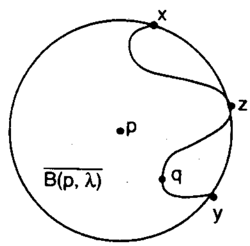

Fig. 6

To finish the proof we have to consider the case $d(p,x)=d(p,y)=d(p,z)=\lambda$. Consider the closed geodesic ball $\overline{B(p,\lambda)}$. Since $\overline{B(p,\lambda)}$ is strictly convex and $x,y,z \in \partial\overline{B(p,\lambda)}$, then there exists a point q of the open geodesic segment (xy) such that $q \neq z$, $q \in \mathrm{Int}(\overline{B(p,\lambda)})$ (see Fig. 6).

From Fig. 4-d we have the $\overline{B(p,\lambda)}$ is a non-convex subset of $\tilde{W}$. The case $\lambda=d(p,x)=d(p,y)<d(p,z)$ is easier than all the above considered cases.

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