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SYMMETRIC R-SPACES

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ABSTRACT

Submanifolds with parallel second fundamental form are defined as extrinsic analogue of locally symmetric manifolds [6, 7]. It follows that all of them are locally invariant under the reflection in the normal space of an arbitrary point. These type of submanifolds are also called symmetric submanifolds [7]. Examples are symmetric *R*-spaces.

Submanifolds with pointwise planar normal sections (P2-PNS) are introduced in [3, 4, 5]. It has shown that spherical submanifolds have P2-PNS property if and only if they must be parellel submanifolds.

In [1] the present author and A. West showed that non-parallel submanifold M has P2-PNS property if and only if It is a hypersurface.

In this article we prove that if M is a symmetric R-space then it must be the orbit of the element Δ such that $ad\,(\Delta))^3 = ad(\Delta)$. We also show that the imbeddings of the symmetric R-spaces of the form f: $M = K/Ko \longrightarrow P$ by $f([k]) = Ad(k)\,\Delta$ have P2-PNS.

1. INTRODUCTION

Let M be a smooth m-dimensional submanifold in (m + d)-dimensional Euclidean space $\mathbf{R}^{m+d}$. For $x \in M$ and a non-zero vector X in $T_x M$ we define the (d + 1)-dimensional affine subspace ξ (x, X) of $\mathbf{R}^{m+d}$ by

$$E(x, X) = \{x + \text{Span}\,\{X, N_x(M)\}\}$$

in a neighbourhood of x. The intersection of $M \cap E$ (x, X) is a regular curve $\gamma: (-\varepsilon, \varepsilon) \longrightarrow M$. We suppose the parameter $t \in (-\varepsilon, \varepsilon)$ is a multiple of the arc-length such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X$. Each choice of $X \in T_x(M)$ yields a different curve which is called the *normal section* of M at x the direction of X where $X \in T_x(M)$ [4]. For such a normal section we can write $\gamma(t) = x + \lambda(t)\,X + N(t)$ where N(t) is the normal part of γ (t).

The submanifold M is said to have *pointwise 2–planar normal sections* (P2–PNS) if each normal section γ the higher order derivatives $\dot{\gamma}(t)$, $\ddot{\gamma}(t)$, $\dddot{\gamma}(t)$ are linearly dependent as vectors in $\mathbf{R}^{m+d}$.

Submanifolds with pointwise 2–planar normal sections have been well studied in the case when M is *spherical* that is; M—S—$\mathbf{R}^{m+d}$.

2. BASIC THEOREM

Let M be an m–dimensional submanifold in (m + d)–dimensional Euclidean space $\mathbf{R}^{m+d}$. Let ∇ and D denote the covariant derivatives of M and $\mathbf{R}^{m+d}$, respectively, Thus D_X is just the directional derivative in the direction X in $\mathbf{R}^{m+d}$. Then for tangent vector fields X, Y and Z over M we have

$$D_X Y = \nabla_X Y + h(X, Y)$$

where h is the second fundamental form of M [3]. We define $\nabla_X h$ as usual by

$$\bar{\nabla}_X(h(Y, Z)) = (\bar{\nabla}_X h)(Y, Z)) + h(\nabla_X Y, Z)) + h(Y, \nabla_X Z)).$$

Then we have the *Gauss* and *codazzi equtions*

$$h(X, Y)) = h(Y, X)$$

and

$$(\bar{\nabla}_X h)(Y, Z)) = (\bar{\nabla}_Y h)(X, Z)) = (\bar{\nabla}_Z h)(X, Y)) = (\bar{\nabla}_X h)(Z, Y))$$
$$= (\bar{\nabla}_Y h)(Z, X)) (\bar{\nabla}_Z h)(Y, X))$$

If $\bar{\nabla} h = 0$ then M is said to have parallel second fundamental form.

Let us write

$$H(X) = h(X, X)$$
$$\nabla H(X) = (\nabla_X h)(Y, Z))$$

so that H, ∇H: T $(M) \longrightarrow$ N (M) are fibre maps whose restriction to each fibre $T_X(M)$ is a homogeneous polynomial map. H is of degree 2 and ∇H is of degree 3 [1].

Proposition 2.1. M has P2–PNS if and only if for each $x \in M$ and each $X \in T_X(M)$ the vectors $H(X)$ and $\nabla H(X)$ in $N_X(M)$ are linearly dependent.

Proof: See [1].

Theorem 2.2. Let M be an m–dimensional submanifold of $\mathbf{R}^{m+d}$. Then M has P2–PNS if and only if

$$\| H \|^2 \nabla H = (H, \nabla H).$$

Proof: See [1].

3. SYMMETRIC R–SPACES

Let g be a semi–simple lie algebra over R and let k be a maximal compact subalgebra of g i.e. a subalgebra of g corresponding to a maximal compact subgroup of the adjoint group g. Let g_c be the complexification of g [2]. Let G_c be the adjoint group g_c; that is

$$G_c = \mathrm{Ad}\,(g_c) = \exp\,(\mathrm{ad}\,(g_c)) \subset GL\,(g_c).$$

Then, as is well–known, there exist a uniquely determined compact form g_u of g_c such that $g \cap g_u = k$, and that letting P denote the orthogonal complement of k in g with respcct to the killing form [10], we have

$$g = k + P$$
$$g_u = k + iP$$

such that

$$[k, k] \subseteq k, \ \ [P, P] \subseteq k, \ \ [k, P] \subseteq P.$$

Let h_p be a maximal abelian subalgebra of P; it can be extended to a Cartan subalgebra h of g; i.e. a maximal subalgebra h of g such that the adjoint representation of any $H \in h$ is semi simple. Then we have

$$h = h_k \cap h_p$$
$$h_k = h \cap k$$
$$h_p = h \cap P.$$

Let h_c be the complexification of h and let

$$g_c = h_c + \sum_{\alpha} g_\alpha \ (\text{where } \alpha \in r)$$

be the corresponding decomposition of g_c, where r denotes the root system of g_c with respect to h_c. Let further h_c be the subspace of h_c over R consisting of all $H \in h_c$ such that $\alpha(H)$ is real for all $\alpha \in r$; then

$$h_0 = ih_k \cap h_p$$

becomes a real Euclidean space with respect to the Killing form, so that we can consider r as a subset of h_0 (i.e. identify $\alpha \in r$ with the uniquely determined element H_a of h_0 such that

$$(H_a, H) = \alpha(H)$$

for all $H \in h_0$, $<,>$ denotes the Killing form) ([10].

Let K be connected subgroup of G_c generated by k and let

$$P = P_0 + m$$

be canonical decomposition for the Lie algebra of K. For $O: = K_0 \in M$ identify $T_0(M)$ [6].

Define

$$K_0 = \{k \in K;\ Ad(k) \triangle = \triangle\},\ \text{where } O \neq \triangle \in P$$

and form the differentiable manifold $M: = K / K_0$.

We can define an embedding

$$f: M: K / K_0 \to P \text{ by } f([k]) = Ad(k) \triangle,\ O \neq \triangle \in P \qquad (3.1)$$

into the Euclidean space with metric given by the Killing form of g.

The differential of f at [e] (e is the identity of g_c) is given by

$$f_* X = ad(X) \triangle \text{ for } X \in m. \qquad (3.2)$$

Definition 3.1. Let M: K / K_0 be a differentiable manifold defined as before. The Riemannian metric induced on M turns M into a Riemannian symmetric space. If

$$ad(\triangle)^3 = ad(\triangle)$$

then M is called *symmetric R–space*, and f its standard imbedding [7].

So $ad(\triangle)^3 = ad(\triangle)$ means there exists an element $O \neq \triangle \in P$ such that $ad(\triangle)$ has eigen values 0, –1, 1 and g admits a decomposition into eigen spaces

$$g = g_0 + g_1 + g_{-1}.$$

For any X, Y $\in$ m we can define the second fundamental form h of M: K / K_0 by

$$h(X, Y) = f_*(X) f_*(Y) \triangle \text{ where } \triangle: = f(0). \qquad (3.3)$$

By (3.1) and (3.3) we have

$$h(X, Y) = ad(X)\, ad(Y) \triangle$$
$$= [X, [Y, \triangle]].$$

Differentiating this at $\triangle$ we have

$$(\bar{\nabla}_Z h)(X, Y) = \{ad(Z)\, ad(X)\, ad(Y) \triangle\}^{\perp} \tag{3.4}$$

This means that for any $X \in m$

$$ad(X) \triangle = [X, \triangle] = X, \tag{3.5}$$

$$ad(X)\, ad(X) \triangle = [X, [X, \triangle]] = h(X, X), \tag{3.6}$$

$$\{ad(X)\, ad(Y) \triangle\{ = [X, [X, \triangle]]]^{\perp} = (\triangle_Z h)(X, Y). \tag{3.7}$$

We have the following;

Proposition 3.2. Let $f\colon M\colon K/K_0 \dashrightarrow P$ be the embedding as before and M be a symmetric R-space. Then $h(X, X)$ and $(\triangle_X h)(X, X)$ are linearly dependent if and only if $[X, [X, \triangle]])]$ and $[X, [X, [X, \triangle]]]^{\perp}$ are linearly dependent.

Proposition 3.3. If $ad(\triangle)^3 = ad(\triangle)$ then for any positive system of generators for the roots $\alpha_1, \alpha_2, \ldots, \alpha_l$ with respect to $\triangle$. There is a unique j such that $\alpha_j(\triangle) = 1$ and other $\alpha_s(\triangle) = 0$, $1 \leq s \leq l$, $s \neq j$.

Proof: Let X_α be a root for a positive root α. Then

$$[H_\alpha, \triangle] = \alpha(\triangle) X_\alpha.$$

Since the eigenvalues of $ad(\triangle)$ are $-1, 0, 1$ we have

$$\alpha(\triangle) = < \triangle, H_a > = -1, 0 \text{ or } 1$$

for every root $\alpha \in r$. Since $\alpha_i(\triangle) \geq 0$ for all simple roots α_i, $i = 1, 2, \ldots, l$ we have $\alpha(\triangle) = 0$ or 1 for every positive root α.

By Kobayashi–Nagano's Lemma [8] there is a unique α_j such that $\alpha_j(\triangle) \neq 0$, and for such an α_j there is a highest root $\theta = \Sigma\, m_i \alpha_i$ such that $\alpha_j(\triangle) = 1$ and $m_j(\triangle) = 1$ and $m_j = 1$.

Definition 3.4. $f\colon M \longrightarrow R^{m+d}$ is an (extrinsic) symmetric submanifold if for every $x \in M$ there is an isometry i of M into itself such that $i(x) = x$ and $f o i = s_x\, o f$, where s_x is a reflection at the normal space through $f(x)$ normal to $f_*(T_x(M))$, and reflects $f(x) + f_*(T_x(M))$ at $f(x)$ [7].

Proposition 3.5. Extrinsic symmetric submanifolds have P2–PNS.

Proof: Let M be a symmetric submanifold and f: $M \longrightarrow \mathbb{R}^{m+d}$ be an isometric immersion. For each $x \in M$ let s_x denote the reflection at the normal space $N_x(M)$ of M at x.

Let γ be a normal section of M at point $x = \gamma(0)$ in the direction of $X = \dot{\gamma}(0) \in T_x(M)$. We have

$$\ddot{\gamma}(0) = h(X, X)$$

$$\dddot{\gamma}(0)^{\perp} = (\overline{\nabla}_X h)(X, X).$$

So by Blomstrom [2] we have

$$(s_x)_*(\overline{\nabla}_X h)(X, X) = (\overline{\nabla}_X h)(X, X).$$

On the other hand, since s_x is affine

$$(s_x)_*(\overline{\nabla}_X h)(X, X) = (\overline{\nabla}_{(s_x)_* X} h)((s_x)_* X. (s_x)_* X).$$
$$= (\overline{\nabla}_{-X} h)(-X, -X) = (\overline{\nabla}_X h)(X, X).$$

Hence $(\nabla_X h)(X, X) = O$ at the point x. So by Theorem 2.1 we get the result...

Proposition 3.6. Standart imbedded symmetric R–spaces are extrinsically symmetric submanifolds.

Proof: See [6)].

Theorem 3.7. If M is a symmetric R–space then M is the orbit of an element $\triangle$ such that $(\mathrm{ad}(\triangle))^3 = \mathrm{ad}(\triangle)$.

Proof: Let g be the semi–simple Lie algebra

$$g = g_0 + g_1 + g_{-1}$$

where

$$[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$$

for $\alpha, \beta \in Z$ such that $g_\mu = \{0\}$ for $\mu \neq \{0\}, \mp 1$.

$$g = k + P$$
$$k = k_0 + m$$

where

$$k = \{X \in g : \rho(X) = X\}$$

$$p = \{X \in g : \rho(X) = -X\}$$

$$k_0 = k \cap g_0$$

$$m = k \cap (g_{-1} \oplus g_1).$$

Let $K = Ad_p(k) = \{\exp \, ad \, (k) \mid_p\} \subset GL(p)$

and set

$$K_0 = \& \, \{k \in K;\ k(\triangle) = \triangle\}.$$

Then K_0 is a closed subgroup of K. Let $K(\triangle)$ be the K–orbit space at $\triangle$. Then by Naitoh [9], $K(\triangle)$ is diffeomorphic to the homogeneous space K / K_0.

The tangent space $T_0(K(\triangle))$ is identified with $[m, \triangle]$.

Since K acts isometrically for $<,>_p$ the orbit space $K(\Delta)$ with the metric induced from $<,>$ is a symmetric space.

Remark 3.8. In [6] Ferus has also proved that if a spherical submanifold has P2–PNS or rather has parallel second fundamental form h. Then it is extrinsically symmetric space.

Corollary 3.9. The imbeddings of the symmetric R–spaces defined as before have P2–PNS.

Proof: Let $M: = K / K_0$ be symmetric R–space and γ be a normal section of M at point $x = \gamma(0)$ in the direction of $\dot{\gamma}(0) = X$. Then by Definition 3.1 we have

$$ad\,(X)^3(\Delta) = ([X, [X, \Delta]]) = [X, \Delta].$$

Combining this with (3.5), (3.7) we have $(\overline{\nabla}_X h)(X, X) = 0$. So by Theorem 2.2. M has P2–PNS.

ÖZET

Paralel ikinci temel forma sahip altmanifoldlar ilk defa Ferus tarafından sınıflandırılmış olup bunlara paralel altmanifoldlar da denir. Simetrik R–uzayları bu tip altmanifold örnekleridir [6 ,7].

Noktasal 2–düzlemsel normal kesitlere (P2–PNS) sahip altmanifoldların [3,4,5] de paralel, [1] de ise paralel olmama hali incelenmiştir.

Bu çalışmada, verilen bir simetrik R–uzayı $M = K / K_0$ için M nin ad $(\triangle))^3$ = ad $(\triangle)$ eşitliğini sağlayan bir $\triangle$ elemanının yörüngesi olduğu ve bunların f: $M = K / K_0 \dashrightarrow P$, f $([k])$ = Ad (k) $\triangle$ şeklindeki gömmeleri P2–PNS şartını sağladığı gösterilmiştir.

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