

ON THE RELATIONS BETWEEN l_p NORMS FOR MATRICES

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ABSTRACT

The l_p norm, l_p operator norm and mixed l_{pq} norm of an $m \times n$ complex matrix $A = (a_{ij})$ are given by

$$|A|_p = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^p \right)^{1/p}$$

and

$$\|A\|_p = \max\{|Ax|_p : x \in \mathbb{C}^n, |x|_p = 1\}$$

and

$$|A|_{pq} = \left\{ \sum_{i=1}^n \left(\sum_{j=1}^m |a_{ij}|^p \right) q/p \right\}^{1/q}$$

respectively [2]. In this study we investigated the relations between l_p norms for matrices and we obtained some results.

STATEMENTS OF MAIN RESULTS

Lemma 1. [1] Let p, q satisfy $1 \leq p, q \leq \infty$. Then for all $x \in \mathbb{C}^n$,

$$|x|_p \leq \lambda_{pq}(n) |x|_q \tag{1}$$

where

$$\lambda_{pq}(n) = \begin{cases} 1 & , p \geq q \\ n^{\frac{1}{p} - \frac{1}{q}} & , p \leq q \end{cases}$$

Theorem 1. Let A be $n \times n$ any complex matrix and let x be an n -vector satisfying $|x|_q = 1$. Then for $1 \leq p, q \leq \infty$

$$|A|_p \leq \xi_{pq}(n) \|A\|_q,$$

where

$$\zeta_{pq}(n) = \begin{cases} n^{\frac{1}{p}} & , p \geq q \\ n^{\frac{2}{p}} - \frac{1}{q} & , p \leq q \end{cases}$$

Proof: Let us denote with a_1, a_2, \dots, a_n the columns of matrix A . Then by the definition of l_p matrix norm, we write

$$\|A\|_p = (|a_1|_p + |a_2|_p + \dots + |a_n|_p)^p.$$

On the other hand, from Lemma 1 and since

$$a_j = Ae_j \quad j = 1, 2, \dots, n$$

(where e_j is a suitable member of the standard basis of C^n .) we have

$$\begin{aligned} \|A\|_p &\leq (\lambda_{pq}(n) |a_1|_q + \lambda_{pq}(n) |a_2|_q + \dots + \lambda_{pq}(n) |a_n|_q)^p \\ &= \lambda_{pq}(n) (|a_1|_q + |a_2|_q + \dots + |a_n|_q)^p \\ &= \lambda_{pq}(n) (|Ae_1|_q + |Ae_2|_q + \dots + |Ae_n|_q)^p \\ &\leq \lambda_{pq}(n) \max_{1 \leq j \leq n} |Ae_j|_q (1, 1, \dots, 1)^p \\ &= \lambda_{pq}(n) n^{\frac{1}{p}} \max_{1 \leq j \leq n} |Ae_j|_q \\ &\leq \lambda_{pq}(n) n^{\frac{1}{p}} \max_{\|x\|_q=1} |Ax|_q \\ &= \lambda_{pq}(n) n^{\frac{1}{p}} \|A\|_q. \end{aligned}$$

Whereas if we consider the definition of $\lambda_{pq}(n)$ in Lemma 1, then

for $p \geq q$, since $\lambda_{pq}(n) = 1$ in this case $\lambda_{pq}(n) \cdot n^{\frac{1}{p}}$ is equal to $n^{\frac{1}{p}}$.

Again for $p \leq q$ since $\lambda_{pq}(n) = n^{\frac{1}{p}} - \frac{1}{q}$, then $\lambda_{pq}(n) \cdot n^{\frac{1}{p}}$

is equal to $n^{\frac{2}{p} - \frac{1}{q}}$.

Thus the theorem is proved.

Theorem 2. Let A be any $n \times n$ complex matrix. Then for $1 \leq p \leq q \leq \infty$

$$\|A\|_p \leq n^{\frac{q-p}{pq}} \|A\|_q$$

Proof: Let z be an n -vector satisfying

$$|z|_p = 1, \quad |Az|_p = \|A\|_p.$$

Denoting $s = \frac{z}{|z|_q}$ we have $|s|_q = 1$. Hence using Lemma 1 we find

$$\begin{aligned} \|A\|_p &= |Az|_p \leq n^{\frac{1}{p} - \frac{1}{q}} |Az|_q \\ &= n^{\frac{q-p}{pq}} |As|_q |z|_q \\ &\leq n^{\frac{q-p}{pq}} |As|_q |z|_p \\ &= n^{\frac{q-p}{pq}} |As|_q \\ &\leq n^{\frac{q-p}{pq}} \max_{|s|_q = 1} |As|_q \\ &= n^{\frac{q-p}{pq}} \|A\|_q. \end{aligned}$$

Thus the proof is completed.

Theorem 3. Let A be $n \times n$ any complex matrix. Then for $1 \leq q \leq p \leq \infty$

$$\|A\|_p \leq \|A\|_q.$$

Proof: From Lemma 1, for $1 \leq q \leq p \leq \infty$ since

$$\|Az\|_p \leq \|Az\|_q$$

Therefore the proof is immediately seen.

Theorem 4. Let A be any $n \times n$ complex matrix. Then for $1 \leq p, q \leq \infty$

$$\|A\|_p \leq n^{\frac{1}{p}} \|A\|_{qp}.$$

Proof: Let us take $B = AE$ where E is $n \times n$ identity matrix. Thus from the product of matrices we have

$$B = (b_{ij}) = \left(\sum_{k=1}^n e_{ik} a_{kj} \right).$$

On the other hand using Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we write

$$|b_{ij}| = \left| \sum_{k=1}^n e_{ik} a_{kj} \right| \leq \left\{ \sum_{k=1}^n |e_{ik}|^p \right\}^{\frac{1}{p}} \left\{ \sum_{k=1}^n |a_{kj}|^q \right\}^{\frac{1}{q}}$$

Thus we obtain

$$\|b_{ij}\|_p^p = \left| \sum_{k=1}^n e_{ik} a_{kj} \right|^p \leq \left(\sum_{k=1}^n |e_{ik}|^p \right) \left(\sum_{k=1}^n |a_{kj}|^q \right)^{\frac{p}{q}}$$

Consequently, we find

$$\begin{aligned} \|B\|_p^p &= \sum_{i,j=1}^n \|b_{ij}\|_p^p \\ &\leq \sum_{i,j=1}^n \left(\sum_{k=1}^n |e_{ik}|^p \right) \left(\sum_{k=1}^n |a_{kj}|^q \right)^{\frac{p}{q}} \\ &= \sum_{i=1}^n \left(\sum_{k=1}^n |e_{ik}|^p \right) \sum_{j=1}^n \left(\sum_{k=1}^n |a_{kj}|^q \right)^{\frac{p}{q}} \\ &= n \cdot \|A\|_{qp}^p \end{aligned}$$

Whereas since $\|B\|_p = \|A\|_p$, we write

$$\|A\|_p \leq n^{\frac{1}{p}} \|A\|_{qp}.$$

So the theorem is proved.

Theorem 5. Let $A = (a_{ij})$ be $n \times n$ complex matrix. Then for $p \geq 1$

$$\|A\|_p^p \leq \sum_{i,j=1}^n \sum_{k=0}^p \binom{p}{k} |b_{ij}|^{p-k} |c_{ij}|^k$$

where $A = B + iC$ and $k = 0, 1, \dots, p$.

Proof: We know that every complex matrix A can be written in the following form:

$$A = B + iC \text{ or for all } 1 \leq i, j \leq n, a_{ij} = b_{ij} + ic_{ij}.$$

On the other hand, since

$$|a_{ij}| \leq |b_{ij}| + |c_{ij}|,$$

for $p \geq 1$ we have

$$|a_{ij}|^p \leq (|b_{ij}| + |c_{ij}|)^p.$$

By Binominal expanded, we obtain

$$\begin{aligned} (|b_{ij}| + |c_{ij}|)^p &= \binom{p}{0} |b_{ij}|^p + \binom{p}{1} |b_{ij}|^{p-1} |c_{ij}| + \dots + \binom{p}{p} |c_{ij}|^p \\ &= \sum_{k=0}^p \binom{p}{k} |b_{ij}|^{p-k} |c_{ij}|^k. \end{aligned}$$

Therefore we get

$$\begin{aligned} \|A\|_p^p &= \sum_{i,j=1}^n |a_{ij}|^p \leq \sum_{i,j=1}^n (|b_{ij}| + |c_{ij}|)^p \\ &= \sum_{i,j=1}^n \sum_{k=0}^p \binom{p}{k} |b_{ij}|^{p-k} |c_{ij}|^k, \end{aligned}$$

and so the theorem is proved.

REFERENCES

- [1] GASTINEL, N.: Linear Numerical Analysis, Hermann, Paris, (1970).
- [2] TASCI, D.: On a Conjecture by Goldberg and Newman, Linear Algebra and Its Applications (to appear).