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A LA MEMOIRE D'ATATÜRK AU CENTENAIRE DE SA NAISSANCE

On The Degree Of Approximation Of Continuous Functions

by

PREM CHANDRA

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Faculté des Sciences de l'Université d'Ankara
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DEDICATION TO ATATÜRK'S CENTENNIAL

Holding the torch that was lit by Atatürk in the hope of advancing our Country to a modern level of civilization, we celebrate the one hundredth anniversary of his birth. We know that we can only achieve this level in the fields of science and technology that are the wealth of humanity by being productive and creative. As we thus proceed, we are conscious that, in the words of Atatürk, "the truest guide" is knowledge and science.

As members of the Faculty of Science at the University of Ankara, we are making every effort to carry out scientific research, as well as to educate and train technicians, scientists, and graduates at every level. As long as we keep in our minds what Atatürk created for his Country, we can never be satisfied with what we have been able to achieve. Yet, the longing for truth, beauty, and a sense of responsibility toward our fellow human beings that he kindled within us gives us strength to strive for even more basic and meaningful service in the future.

From this year forward, we wish and aspire toward surpassing our past efforts, and with each coming year, to serve in greater measure the field of universal science and our own nation.

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ABSTRACT

In this paper the author has obtained the degree of approximation of a 2π-periodic function f of the class Lip α, $0<\alpha\leq 1$, by Eulermeans of its Fourier Series. He has further shown that this degree of approximation is best possible in certain sense.

1. Let f be 2π -periodic and L-integrable over $[-\pi,\pi]$. The Fourier series associated with f at a point x is

$$(1.1) \qquad \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

A function $f \in$ Lip α $(\alpha>0)$ if

$$(1.2) \qquad f(x+h) - f(x) = O(|h|^{\alpha}).$$

Let $\{s_n\}$ be a sequence of partial sums of the given series $\sum_{n=0}^{\infty} c_n$, where $s_n = c_o + c. + \dots + c_n$. Then (E, q) $(q > 0)$ –means of $\{s_n\}$ are defined by ([4], p. 180)

$$(1.3) \qquad (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \; .$$

The (E, q) means for $q > O$ are regular ([4], p. 179). Throughout this paper K's will denote positive constants.

2. INTRODUCTION. The following theorem on the degree of approximation of a function f, belongingto the class Lip α by the (C, δ) - means of its Fourier series is proved by Alexits ([1], p. 301):

Theorem A. If a periodic function f belongs to the class Lip α, then the degree of approximation of (C, δ)-means of its Fourier series for $0 < \alpha \leq \delta \leq 1$ is

$$(2.1) \quad \max_{0 \leq x \leq 2\pi} | f(x) - \sigma_n^{\delta}(x) | = \begin{cases} O(n^{-\alpha}), \text{ for } 0 < \alpha < \delta \leq 1; \\ O(n^{-\alpha} \log n) \text{ for } 0 < \alpha \leq \delta \leq 1, \end{cases}$$

where $\sigma_n^{\delta}(x)$ are the (C, δ) – means of the partial sums of (1.1).

By extending the above result, the present author [2] proved the following theorem:

Theorem B. The degree of approximation of a periodic function f with period 2π and belonging to the class Lip α, by (R, p_n)-means of its Fourier series is given by

$$(2.2) \quad \max_{0 \leq x \leq 2\pi} | f(x) - T_n(x) | = \begin{cases} O\{(\frac{p_n}{P_n})^{\alpha}\}; \ 0 < \alpha < 1 \\ O\{\frac{p_n}{P_n} \log \frac{P_n}{p_n}\}; \ \alpha = 1, \end{cases}$$

where (R, p_n) – means are regular and $0 < p_n \uparrow$ (monotonic increasing) with $n \geq n_0$ and $T_n(x)$ are (R, p_n) – means of the partial sums of (1.1).

It may be observed that the means generating sequences in the above theorems are involved either implicitiy or explicitly in the order of approximation. In this paper we obtain the following theorem which provides the degree of approximation free from the means generating sequence:

THEOREM. The degree of approximation of a periodic function f with period 2π and belonging to the class of Lip α, $0 < \alpha \leq 1$, by Euler-means of its Fourier series is given by

$$(2.3) \quad \max_{0 \leq x \leq 2\pi} | f(x) - t^q_n(x) | = O(n^{-\frac{1}{2}\alpha}),$$

where $t^q_n(x)$ are (E, q) $(q > 0)$ –means of the partial sums of (1.1). However the inequality (2.3) is best possible in the sense that there exists a positive constant K such that

$$(2.4) \quad \max_{0 \leq x \leq 2\pi} | f(x) - t^q_n(x) | \geq K n^{-\frac{1}{2}\alpha} \quad (n > n_0)$$

3. For the proof of the theorem we shall require the following lemmas:

LEMMA 1. Let $0 < t \leq \pi$. Then

$$(1+q)^{-n}\{1+q^2+2q\cos t\}^{\frac{n}{2}} < \exp\{-2nqt^2/(\pi(1+q))^2\}.$$

PROOF. We have

$$(1+q)^{-n}(1+q^2+2q\cos t)^{\frac{1}{2}n} = \left\{1-\left(\frac{2\sqrt{q}}{1+q}\sin\tfrac{1}{2}t\right)^2\right\}^{\frac{1}{2}n}$$

$$= \left[\cos\left\{\sin^{-1}\left(\frac{2\sqrt{q}}{1+q}\sin\tfrac{1}{2}t\right)\right\}\right]^n$$

$$< \left\{\cos\left(\frac{2t\sqrt{q}}{\pi(1+q)}\right)\right\}^n$$

since

$$\sin^{-1}\frac{2\sqrt{q}}{1+q}\sin\tfrac{1}{2}t > \frac{2\sqrt{q}}{1+q}\sin\tfrac{1}{2}t$$

$$\geq \frac{2\sqrt{q}}{1+q}\,\frac{t}{\pi} \quad (0 < t \leq \pi),$$

where $\left\{\dfrac{\sin\theta}{\theta}\right\}$ is decreasing in $0 < \theta < \frac{1}{2}\pi$ and its minimum value in $(0, \frac{1}{2}\pi)$ is $\dfrac{2}{\pi}$ so that $\sin\theta \leq \dfrac{2}{\pi}\theta$.

However, for $0 < t \leq \pi$,

$$0 < \frac{2t\sqrt{q}}{\pi(1+q)} < \tfrac{1}{2}\pi$$

therefore, we have (see Hardy [4], p. 364)

$$\cos\frac{2t\sqrt{q}}{\pi(1+q)} < \exp\{-2qt^2/(\pi(1+q))^2\}$$

and hence the proof of Lemma 1 follows.

LEMMA 2. Let $0 < t \leq \frac{4}{\pi}\beta$, where β is strictly less than $\frac{1}{2}\pi$. Then

$$(1+q)^{-n}(1+q^2+2q\cos t)^{\frac{n}{2}} > \exp\{-nqt^2\pi^2\sec^2\beta \,/\, (8(1+q)^2)\}$$

PROOF. We have

$$\log((1-y)^{-1}) < y/(1-y) \qquad (0 < y < 1).$$

Therefore, on replacing y by $\sin^2\theta$ $(0 < \theta \leq \beta < \frac{1}{2}\pi)$ in the above inequality and simplifying, we get

$$\log \sec^2\theta < \tan^2\theta$$
$$< \theta^2\sec^2\beta$$

or

$$\cos\theta > \exp\{-\tfrac{1}{2}\theta^2\sec^2\beta\}.$$

And, as in Lemma 1,

$$(1+q)^{-n}(1+q^2+2q\cos t)^{\frac{1}{2}n} = (\cos z)^n,$$

where

$$z = \sin^{-1}\left(\frac{2\sqrt{q}}{1+q}\sin\tfrac{1}{2}t\right) = \sin^{-1}\left(\frac{2}{\pi}\,\frac{\pi\sqrt{q}\sin\frac{1}{2}t}{(1+q)}\right)$$

$$\leq \frac{\pi\sqrt{q}}{1+q}\sin\tfrac{1}{2}t$$

$$\leq \frac{\pi}{2}\,\frac{q^{\frac{1}{2}}}{1+q}\,t$$

$$\leq \beta\left(0 < t \leq \frac{4}{\pi}\beta\right).$$

Hence

$$\cos z \geq \cos\{\pi q^{\frac{1}{2}} t/2(1+q)\}$$

$$> \exp\left\{-\frac{\pi^2}{2}\,\frac{qt^2\sec^2\beta}{4(1+q)^2}\right\}$$

from which we may follow the proof of Lemma 2.

4. PROOF OF THE THEOREM. To simplify the proof of the theorem, we use the following notations:

$$P(q,t,n) = (1+q)^{-n}(1+q^2+2q\cos t)^{\frac{1}{2}n}$$
$$Q(q,t) = \tan^{-1}\{\sin t/(q+\cos t)\}$$
$$a = 2q/(\pi(1+q))^2$$
$$b = q\pi^2/(2(1+q))^2.$$

We have

$$s_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\Phi_x(t)\,dt,$$

where

$$\Phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

Then (E, q) – transform of $\{s_n(x) - f(x)\}$ will be given by

$$(1+q)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\{s_k(x) - f(x)\}$$

which equals to

$$t_n^q(x) - f(x).$$

Therefore

$$t_n^q(x) - f(x) = \frac{1}{\pi(1+q)^n}\int_0^{\pi}\frac{\Phi_x(t)}{\sin\frac{1}{2}t}\left\{\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sin(k+\tfrac{1}{2})t\right\}dt,$$

and spliting up the integral $\int_0^{\pi}$ into the sub-integrals $\int_0^{\frac{1}{\sqrt{n}}}$ and $\int_{\frac{1}{\sqrt{n}}}^{\pi}$ and denoting them respectively by I_1 and I_2, we obtain that

$$|f(x) - t_n^q(x)| \le |I_1| + |I_2|.$$

Now

$$\max_{0\le x\le 2\pi}|I_1| \le \max_{0\le x\le 2\pi}\int_0^{\frac{1}{\sqrt{n}}}\frac{|\Phi_x(t)|}{\sin\frac{1}{2}t}\left\{(1+q)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right\}dt$$

$$= \max_{0\leq x\leq 2\pi} \int_{0}^{\frac{1}{\sqrt{n}}} \frac{|\Phi_x(t)|}{\sin \frac{1}{2} t} dt$$

$$= O \left\{ \int_{0}^{\frac{1}{\sqrt{n}}} t^{\alpha-1} dt \right\}$$

$$= O\,(n^{-\frac{1}{2}\alpha})$$

and, proseeding as in Chandra [3]; (3.2), we obtain that

$$|I_2| \leq \int_{\frac{1}{\sqrt{n}}}^{\pi} \frac{|\Phi_x(t)|}{\sin \frac{1}{2} t} \left| P(q,t,n) \sin \left\{ \tfrac{1}{2} t + nQ(q,t) \right\} \right| dt$$

$$\leq \int_{\frac{1}{\sqrt{n}}}^{\pi} \frac{|\Phi_x(x)|}{\sin \frac{1}{2} t} P(q,t,n)\, dt$$

$$= O \left\{ \int_{\frac{1}{\sqrt{n}}}^{\pi} \frac{|\Phi_x(t)|}{\sin \frac{1}{2} t} \exp(-ant^2)\, dt \right\}$$

(by Lemma 1)

$$= O \left\{ \frac{1}{n} \int_{\frac{1}{\sqrt{n}}}^{\pi} \frac{|\Phi_x(t)|}{t \sin \frac{1}{2} t} \frac{\partial}{\partial t} \left\{ -\exp(-ant^2) \right\} dt \right\}$$

Therefore

$$\max_{0\leq x\leq 2\pi} |I_2| = O \left\{ \frac{1}{n} \int_{\frac{1}{\sqrt{n}}}^{\pi} t^{\alpha-2} \left\{ -\frac{\partial}{\partial t} \exp(-ant^2) \right\} dt \right\}$$

$$= O \left\{ n^{-(\alpha-2)/2-1} \int_{\frac{1}{\sqrt{n}}}^{\pi} \left\{ -\frac{\partial}{\partial t} \exp(-nt^2 a) \right\} dt \right\}$$

$$= O\,(n^{-\frac{1}{2}\alpha}).$$

Thus combining the results, as obtained above, we get

$$\max_{0\leq x\leq 2\pi} \quad | f(x) - t_n^q(x) | = O(n^{-\frac{1}{2}\alpha}) + O(n^{-\frac{1}{2}})$$

$$= O(n^{-\frac{1}{2}\alpha}).$$

This completes the proof of (2.3).

To prove (2.4), we suppose that δ is a small number less than $\pi/4$. Then we write

$$\pi | f(x) - t_n^q(x) | = | \int_0^{\pi} \frac{\Phi_x(t)}{\sin \frac{1}{2} t} \{(1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin (k+\tfrac{1}{2})t\} \, dt |$$

$$= | \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi} \right) \left(\frac{\Phi_x(t)}{\sin \frac{1}{2} t} \left\{ (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} . \right.\right.$$

$$\left.\left. . \sin (k+\tfrac{1}{2}) t \right\} dt \right) |$$

$$= | J_1 + J_2 + J_3 | \text{ (suppose)}$$
$$\geq | J_2 | - | J_1 | - | J_3 |.$$

Proceeding as in $| I_1 |$, we get

$$\max_{0\leq x\leq 2\pi} \quad | J_1 | < K_1 n^{-\alpha}$$

and proceeding as in $| I_2 |$, we get

$$\max_{0\leq x\leq 2\pi} \quad | J_3 | = O \left\{ \frac{1}{n} \int_{\delta}^{\pi} t^{\alpha-2} \left\{ - \frac{\partial}{\partial t} \exp(-ant^2) \right\} dt \right\}$$

$$= O \left\{ \frac{1}{n} \int_{\delta}^{\pi} \left\{ - \frac{\partial}{\partial t} \exp(-ant^2) \right\} dt \right\}$$

$$= O \left(\frac{1}{n} \right),$$

therefore

$$\max_{0\leq x\leq 2\pi} \quad | J_3 | < K_3/n.$$

Now

$$| J_2 | \geq | 2 \int_{\frac{1}{n}}^{\delta} \Phi_x(t) \, t^{-1} P(q,t,n) \sin \{ \tfrac{1}{2} t + nQ(q,t) \} \, dt |$$

$$- \left| \int_{\frac{1}{n}}^{\delta} \Phi_x(t) \; \{\operatorname{cosec} \tfrac{1}{2} t - 2/t\} \; P(q,t,n) . \right.$$

$$\left. . \sin \{\tfrac{1}{2} t + nQ(q,t)\} \, dt \right|$$

$$= 2 \; |J_{2,1}| - |J_{2,2}|, \text{ say.}$$

However

$$\max_{0 \leq x \leq 2\pi} \; |J_{2,2}| = O \left\{ \int_{\frac{1}{n}}^{\delta} t^{1+\alpha} \; P(q,t,n) \, dt \right\}$$

$$= O \left\{ \frac{1}{n} \int_{\frac{1}{n}}^{\delta} t^{\alpha} \left\{ - \frac{\partial}{\partial t} \exp(-ant^2) \right\} dt \right\}$$

(by Lemma 1)

$$= O \left(\frac{1}{n} \right)$$

so that

$$|J_{2,2}| < K_{2,2}/n.$$

Now, by the hypothesis, there exists a constant $K_4 > 0$ such tahat

$$- K_4 t^{\alpha} < \Phi_x(t) < K_4 t^{\alpha}$$

therefore

$$K_4 t^{\alpha} < \Phi_x(t) + 2 \; K_4 t^{\alpha}$$

and hence

$$|J_{2,1}| > \left| \int_{\frac{1}{n}}^{\delta} \frac{\Phi_x(t) + 2K_4 t^{\alpha}}{t} \; P(q,t,n) \sin \left(\tfrac{1}{2} t + nQ(q,t)\right) dt \right|$$

$$- 2 \; K_4 \left| \int_{\frac{1}{n}}^{\delta} t^{\alpha-1} \; P(q,t,n) \sin \left(\tfrac{1}{2} t + nQ(q,t) \right) dt \right|$$

$$= |J_{2,1,1}| - 2 \; K_4 |J_{2,1,2}|, \text{ say.}$$

However $t^{\alpha-1}$ $P(q,t,n)$ is positive and non-increasing in $\frac{1}{n} \leq t \leq \delta$, therefore by the second mean value theorem

$$J_{2,1,2} = n^{1-\alpha} P(q, \tfrac{1}{n}, n) \int_{\frac{1}{n}}^{\delta'} \sin \left(\tfrac{1}{2} t + nQ(q,t)\right) dt$$

$$(n^{-1} \leq \delta' \leq \delta)$$

$$= O\,(n^{-\alpha})$$

so that

$$\max_{0\leq x\leq 2\pi} \quad 2\ K_4 \,|\, J_{2,1,2} \,| \,<\, K_{2,1,2}n^{-\alpha}.$$

And, by the first mean value theorem

$$|\, J_{2,1,1} \,| = \underset{(\frac{1}{n}\leq t'\leq \delta)}{|\, \sin\,(\tfrac{1}{2}\, t' + nQ\,(q,t')\,|} \,\left| \int_{\frac{1}{n}}^{\delta} \frac{\Phi_x(t) \,+ 2K_4t^{\alpha}}{t}\, P\,(q,t,n)\; dt \right.$$

$$\geq |\, \sin\ (\tfrac{1}{2}\ t' + nQ\ (q,t')\ )\ |\ \int_{\frac{1}{\sqrt{n}}}^{\delta} \frac{\Phi_x(t) \,+\, 2K_4t^{\alpha}}{t}\, P\,(q,t,n)\; dt$$

$$> K_4\ |\ \sin\ \{\ \tfrac{1}{2}\ t' \,+ nQ\ (q,t')\ \}\ |\ \int_{\frac{1}{\sqrt{n}}}^{\delta} t^{\alpha-1}P\,(q,t,n)\; dt$$

$$> K_4\ |\ \sin\ \{\ \tfrac{1}{2}\ t' \,+ nQ\ (q\ ,t')\ \}\ |\ \int_{\frac{1}{\sqrt{n}}}^{\delta} t^{\alpha-1}\, \exp\,(-bnt^2)\; dt$$

(by Lemma 2)

$$\geq\ K_{2,1,1}n^{-\frac{1}{2}\alpha},$$

where the constant $K_{2,1,1}$ depends upon $\sin\ \{\ \frac{1}{2}\ t' \,+ nQ\ (q,t')\ \}$ and other parameters. However the integeral $J_{2,1,1}$ is not zero, therefore the constant $K_{2,1,1}$ is positive.

Now, collecting the results we get

$$\max_{0\leq x\leq 2\pi} \quad \pi\ |\ f\,(x) - t_n^q(x)\ |\ \geq K_{2,1,1}n^{-\frac{1}{2}\alpha} - (K_1 + K_{2,1,2})\ n^{-\alpha}$$

$$- (K_3 + K_{2,2})\ /n$$

$$\geq\ K_{2,1,1}n^{-\frac{1}{2}\alpha} - (K_1 \ +\ K_3\ +\ K_{2,2}\ +\ K_{212})\ /n$$

$$= n^{-\frac{1}{2}\alpha}\Big[K_{2,1,1} - (K_1 + K_3 + K_{2,2} + K_{2,1,2})\ /n^{1-\frac{\alpha}{2}}\Big].$$

And, for any given constant K', such that

$$K_{2,1,1} - K' > 0$$

we can find a positive number $n_0 = n_0 (K')$ such that

$$(K_1 + K_3 + K_{2,2} + K_{2,1,2})n^{\frac{1}{2}\alpha-1} \leq K' \text{ for } n > n_0.$$

And, hence writing K for $\frac{1}{\pi}(K_{2,1,1} - K')$, we get

$$\max_{0 \leq x \leq 2\pi} | f(x) - t_n^{\alpha}(x) | \geq K n^{-\frac{1}{2}\alpha} (n > n_0)$$

This completes the proof of the theorem.

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