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A Characterization of Inclined Curves in Enclidean n-Space

by

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ABSTRACT

We define the *harmonik curvatures* H_i, $1 \leq i \leq n\text{-}2$, of a curve X in n-dimensional Euclidean space E^n. We generalize the *inclined curves* (Böschungslinien) of E^3 to E^n and then give a characterization for the inclined curves in E^n:

A curve X is an inclined curve $\Longleftrightarrow \sum_{i=1}^{n-2} H^2_i = \text{constant}.$

I. Basic Concepts

Basic concepts for this paper are the summaries of [2] and [4].

Definition I.1:

In E^n, n-dimensional Euclidean space, a curve is an image of a diffeomorphism

$$X : I \longrightarrow E^n,$$

where I is an open segment of a straight line.

Definition 1.2:

Let $X : I \longrightarrow E^n$ be a parametrized curve with parameter t.

Let J be another interval with parameter s, and let

$$J \xrightarrow{\ Y\ } I \xrightarrow{\ X\ } E^n$$

where Y has a nonvanishing Jacobian. Then XoY is a parametrized curve with parameter s. The curve XoY is called a reparameterization of the curve X.

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Definition I.3:

Let $X: I \longrightarrow E^n$ be a parametrized curve with parameter s. The parameterization X is called the arc-length parameter if $X_* \left(\frac{\partial}{\partial s}\right)$ has length one in $T_{E^n}(X(s))$ for all $s \in I$; where $X_* \left(\frac{\partial}{\partial s}\right)$ is the tangent vector to the curve X and $T_{E^n}(X(s))$ denotes the tangent space of E^n at the point $X(s) \in E^n$.

We need the following well-known theorem [1].

Theorem I.1:

A parameterized curve can always be reparameterized by arac- length parameter.

Theorem I.1 says that, in general, we can have the arc-length parameterized curve X(s) with arc-length parameter s as a parameterized curve in E^n.

For I, an open interval in the real line $\mathcal{R}$, we shall interpret this liberally to include not only the usual type of open interval $a < s < b$ (a, b real numbers), but also the types of $a < s$ (a half-line to $-\infty$), and also the whole real line. Henceforth we doneto an arc-length parameterized curve of E^n by a map

$$X : I \longrightarrow E^n$$

which is a C^∞ parameterization by arc-length.

We assume that each point X(s), of the curve X, the derived vectors

$$\{X'(s), X''(s), \ldots, X^{(r)}(s)\},$$

are linearly independent, where

$$X'(s) = \frac{dX}{ds}(s), \; X''(s) = \frac{d^2X}{ds^2}(s), \ldots, X^{(r)}(s) = \frac{d^rX}{ds^r}(s).$$

Therefore there exists an algorithm, called the Gram-Schmidt process, for converting the vectors

$$X'(s), \ldots, X^{(r)}(s)$$

into an orthonormal basis

$$\{V_1(s),\ V_2(s),,\ \ldots,\ V_r(s)\ \}$$

of the tangent space T_{E^n} $(X(s))$ of E^n at the point $X(s) \in E^n$. This system is called the Frenet r-handed (or r-frame) of the curve X at the point X(s) [2].

If we denote the inner product (dot product) $E^n \times E^n \longrightarrow \mathcal{R}$ over E^n by $< , >$ we have that

$$< V_i , V_j > = \delta_{ij}$$

and then the derivatives of the frame vectors satisfy the following Frenet Equations:

$$\text{(I.1)} \qquad \begin{cases} V'_i = -k_{i-1} V_{i-1} + k_i V_{i+1} , 2 \leq i \leq r\text{-}1 \\ V'_1 = k_1 V_2 \\ V'_r = - k_{r-1} V_{r-1} \end{cases}$$

where $k_i = k_i$ (s), $1 \leq i \leq r-1$, is the curvature, with order i, of the curve X, at its point X (s) [2].

II. Inclined Curves And Its Characterization

a) The Inclined Curves In E^n.

In E^n, we give a definition of inclined curves (böschungslinien) as a generalization of their definition, in E^3 which is given by E. Müller [3]:

Definition II. I:

Let $X{:}I \longrightarrow E^n$ be a curve in E^n with the arc-length parameter s and u be a unit constant vector of E^n. For all $s \in I$, if

$$\text{(II.1)} \qquad < X'(s) , u> = \cos\varphi = \text{constant} , \varphi \neq \frac{\pi}{2}$$

then the curve is called an inclined curve in E^n; where X′(s) is the unit tangent vector to the curve X at its point X(s), and φ is a constant angle between the vectors X′ and u.

In our generalized Definition II. 1, we add the condition that $\varphi \neq \frac{\pi}{2}$ which is not belong to the definition given by E. Müller [3]. If $\varphi = \pi / 2$ then every curve of an hyperplane is an inclined curve, so every curve in E^n can be an inclined curve in E^{n+1}. In this case the characterization for the inclined curves is obvious, so we do not include the special case $\varphi = \pi / 2$.

b) Harmonic Curvatures Of A Curve in E^n.

Now to characterize that a curve in E^n to be an inclined curve we define the concept of the harmonic curvatures, for a curve in E^n, which is known, in the case $n = 3$, as the ratio

$$\frac{\text{first curvature}}{\text{second curvature}} .$$

Definition II. 2:

Let $X: I \longrightarrow E^n$ be a curve in E^n with an arc–lenght paramater s and u be the unit constant vector. Let

$$(V_1(s), \ldots, V_r(s) \; ; \; X(s)) \; , \; 3 \leq r \leq n$$

be the Frenet r-frame of X at its point X(s). If the angle, between $X'(s)$ and u, is $\varphi = \varphi(s)$ we define the function

$$H_i : I \longrightarrow \mathbb{R} \quad , \; 3 \leq i \leq r\text{-}2$$

by

$$< V_{i+2}(s) \, , \, u > = H_i(s) \cos\varphi$$

as the harmonic curvature, with order i, of the curve X at its point X(s). We define also

$H_o = 0$.

If the curve X is an inclined curve then we give the following theorem which gives the relations of the curvatures H_i with each other.

Theorem II.1:

Let $X: I \longrightarrow E^n$ be an inclined curve in E^n with an arc-length parameter s, $k_i(s)$ be the curvature of X, with order i, $\sigma_i = 1/k_i(s)$ and $H_j(s)$, $1 \leq j < r\text{-}1$, be the harmonic curvature with order j. Then we have

$$H_1 = k_1/k_2 \tag{II.3}$$

$$H_j = [H'_{j-1} + H_{j-2} \, k_j] \, \sigma_{j+1} \; , \; 2 \leq j \leq n-2. \tag{II.4}$$

Proof:

Let u be the constant unit vector such that

$$<X'(s), u> = \cos\varphi = \text{constant, for } \forall s \in I.$$

Differentiating this, with respect to s, we obtain that

(II.5) $$< V'_1 , u > = 0$$

or

(II. 6) $$k_1 < V_2 , u> = 0 ,$$

where $k_1 \neq 0$, in other case all of the other curvatures must be zero [2]. Then (II. 6) gives us that

(II. 7) $$<V_2 , u> = 0.$$

Again, differentiating (II. 7), with respect to s, and using the Equations (I. 1) we obtain

(II. 8) $$<-k_1 V_1 + k_2 V_3 , u> = 0$$

and so from (II.2) we have

$$\cos\varphi\, H_1 = <V_3 , u> = \frac{k_1}{k_2} \cos\varphi$$

or

$$H_1(s) = k_1(s) \,/\, k_2(s)$$

which is the only harmonic curvature of a curve in E^3 . This is the first one in the space E^n, $n > 3$ and we have higher harmonic curvatures.

For the higher harmonic curvatures let differentiate the Equation (II.2), with respect to s, then we have

$$<-k_i V_i + k_{i+1} V_{i+2} , u> = H'_{i-1} \cos\varphi$$

or

$$-k_i <V_i, u> + k_{i+1} <V_{i+2} , u> = H'_{i-1} \cos\varphi$$

and then

$$H_i = [H'_{i-1} + H_{i-2} k_i]\, \sigma_{i+1} , \quad 2 \leq i \leq n-2$$

which completes the proof. |||||||

Remark:

If we take $i = r - 2$, in (II. 2), we obtain

(II.11) $H_{r-2} \cos\varphi = \langle V_r , u \rangle$

and differentiating it, with respect to s, we have

(II.12) $-H_{r-3} = H'_{r-2} \sigma_{r-1}$.

On the other hand for $j = r - 1$, (II. 4) gives us

(II.13) $H_{r-1} = [H'_{r-2} + H_{r-3} k_{r-1}] \sigma_r$.

Replacing (II. 12) in (II. 13) we see that

(II.14) $H_{r-1} = 0$.

Since we we have r- handed Frenet frame at every point $X(s) \in X$ it must be that

(II.15) $k_r (s) = 0$,

and so (II. 14) says that H_{r-1} (s) is indefinite in this case.

According to Theorem (II. 1) the functions H_i are not dependent on the choices of the vector u.

c) A Characterization For The Inclined Curves in E^n.

An inclined curve in E^n has a characterization in terms of its harmonic curvatures. We give this characterization in the following theorem.

Theorem II. 2:

Let $X: I \longrightarrow E^n$ be a curve in E^n such that there exists a Frenet n-frame, at its every point X(s). If s is the arc-length parameter and H_j, $1 \leq j \leq n\text{-}2$

are the harmonic curvatures at the point X(s) of th ecurve X then we have:

The curve X is an inclined curve $\Longleftrightarrow$

(II.16) $\sum_{j=1}^{n-2} H^2_j (s) = \text{constant}.$

Proof. (Necessity): Let $X : I \longrightarrow E^n$ be an inclined curve. Then there exists a constant unit vector u for the curve X such that

$$\langle X' (s), u \rangle = \cos\varphi = \text{const., for } \forall s \in I.$$

Thus, according to the basis, which is Frenet n-frame

$$(V_1, V_2, \dots, V_n : X(s))$$

at the point $X(s) \in X$, we can express the unit vector u as follows

(II.17) $$u = \sum_{i=1}^{n} \langle V_i(s), u \rangle V_i(s) .$$

Replacing (II. 1), (II, 2) and (II. 7) in (II. 17) the condition

(II. 18) $$\| u \| = 1$$

gives us that

$$\cos^2 \varphi + \sum_{i=1}^{n-2} H_i^2(s) \cos^2 \varphi = 1$$

or since $\varphi \neq \pi / 2$ is a constant angle

(II.19) $$\sum_{i=1}^{n-2} H_i^2(s) = tg^2\varphi = \text{constant}$$

which completes the proof of the necessity.

Sufficiency: Let $X: I \longrightarrow E^n$ be a curve in E^n such that its harmonic curvatures satisfy the relation

$$\sum_{i=1}^{n-2} H^2_i(s) = a \text{ (constant)}$$

at every point $X(s) \in X$. Then we can find an angle $\varphi \in (0, 2\pi)$ such that $tg^2 \varphi = a$.

Using the above notations, let define a vector u as follows

(II.20) $$u = \cos\varphi V_1(s) + \sum_{i=3}^{n} H_{i-2}(s) \cos\varphi\, V_i(s).$$

The vector u is a constant vector:

From (II. 20) we may write that

(II.21) $$\frac{1}{\cos\varphi} \frac{du}{ds} = V'_1(s) + \sum_{i=3}^{n} H'_{i-2}(s) V_i(s) + \sum_{i=3}^{n} H_{i-2}(s) V_i'(s)$$

and where taking $j = i - 2$ and using (I. 1) we obtain

(II.22) $$V'_{j+2}(s) = -k_{j+1}(s) V_{j+1}(s) + k_{j+2}(s) V_{j+3}(s).$$

From (II.4) we can calculate

(II.23) $H'_j(s) = -k_{j+1}(s)\ H_{j-1}(s) + k_{j+2}(s)\ H_{j+1}(s)$

and replacing it in (II.22) we have that

$$\text{(II.24)}\quad \frac{1}{\cos\varphi}\ \frac{du}{ds} = k_1 V_2 + \sum_{j=1}^{n-2} [-k_{j+1}\ H_{j-1} + k_{j+2}\ H_{j+1}]\ V_{j+2}$$

$$+ \sum_{j=1}^{n-2} H_j\ [-k_{j+1}\ V_{j+1} + k_{j+2}\ V_{j+3}].$$

Since we know that

$H_1 = k_1/k_2$, $V_2 = k_1 \cdot V_2$, $H_0 = 0$ and $H_{n-1} = 0$

(II.24) reduces to

$$\frac{1}{\cos\varphi}\quad \frac{du}{ds} = 0$$

which says that u is constant vector.

u is a unit vector: Indeed if we calculate the norm of u from (II.20) we see that

$$||\ u\ || = 1.$$

On the other hand from (II.20) we have that

$$\langle V_1(s)\ ,\ u\rangle = \cos\varphi = \text{constant}$$

which completes the proof of the sufficiency. |||||||

d) Special Cases:

In the case n = 3 since we have just one harmonic curvature which is

$$H_1(s) = k_1\ (s)\ /\ k_2\ (s)$$

the condition (II.19) reduces to

$$H^2_1 = k^2_1\ /\ k^2_2 = t^2g\varphi$$

or

$$\text{(II.25)}\qquad \frac{k_1(s)}{k_2(s)} = tg\varphi = \text{constant}.$$

which is well-known in the classical books about the differential geometry, for example [3].

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ÖZET

E^n n-boyutlu Öklid uzayında bir X eğrisinin H_i, $1 \leq i \leq n-2$, ***harmonik eğrilik***lerini ve ayrıca ***eğilim çizgileri***ni tanımladık. Sonra bu çizgilerin karakterizasyonunu H_i harmonik eğrilikleri cinsinden verdik:

$$X \text{ eğrisi bir eğilim çizgisidir} \Longleftrightarrow \sum_{i=1}^{n-2} H_i^2(s) = \text{sabit}.$$

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