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**On The Mean Values Of an Entire Function represented  
By a Dirichlet Series II**

by

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## On The Mean Values Of an Entire Function represented By a Dirichlet Series II

JODH PAL SINGH

### SUMMARY

In this note we prove a theorem which gives us information as to how the functions  $\log I_{\delta}(\sigma)$  and  $\log J_{\delta,k}(\sigma)$  grow relative to each other as  $\sigma \rightarrow \infty$ .

Theorem. Let  $f(s)$  be an entire function represented by a Dirichlet series, then

$$\lim_{\sigma \rightarrow \infty} \frac{\log I_{\delta}(\sigma)}{\log J_{\delta,k}(\sigma)} \leq \bar{\lambda} \{ (1 + k/\bar{\lambda}) \}^{(1+\bar{\lambda}/k)} \quad (\bar{\lambda} > 0)$$

$$\leq 1, \quad (\bar{\lambda} = 0)$$

where  $\bar{\lambda} = \lim_{\sigma \rightarrow \infty} \frac{\log \log I_{\delta}(\sigma)}{\sigma}$

1. In the usual notation,

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \quad (s = \sigma + it), \quad 0 < \lambda_n < \lambda_{n+1} \quad (n \geq 1)$$

$$\lim_{n \rightarrow \infty} \lambda_n = \infty$$

is an entire function in the sense that the Dirichlet series representing it, is absolutely convergent for all finite  $s$  and possesses

$$\overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log M(\sigma)}{\sigma} = \rho$$

$$\lambda$$

where  $0 \leq \lambda, \rho \leq \infty$ , and  $M(\sigma)$  have their usual meanings.

The mean values of  $f(s)$  are defined as follows:

$$(1.1) \quad I_{\delta}(\sigma) = \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^{\delta} dt \right\}^{\frac{1}{\delta}}, \quad (\delta > 0)$$

and

$$(1.2) \quad J_{\delta,k}(\sigma) = \exp \left\{ \frac{1}{e^{k\sigma}} \int_0^{\sigma} \log I_{\delta}(x) e^{kx} dx \right\} \quad 0 < k < \infty.$$

The mean value (1.1) for  $\delta = 2$  was defined by Hadamard [1], and it is known that

$$\frac{\bar{\rho}}{\lambda} = \overline{\lim}_{\sigma \rightarrow \infty} \frac{\log \log I_{\delta}(\sigma)}{\sigma} \leq \frac{\rho}{\lambda} \quad [2]$$

Then, following Shah [3], there exists a lower proximate order  $\bar{\lambda}(\sigma)$  relative to  $\log I_{\delta}(\sigma)$ , satisfying the following conditions:

- (i)  $\lim_{\sigma \rightarrow \infty} \bar{\lambda}(\sigma) = \bar{\lambda}$ ,
- (ii)  $\lim_{\sigma \rightarrow \infty} \sigma \bar{\lambda}'(\sigma) = 0$ ,
- (iii)  $\log I_{\delta}(\sigma) \geq \exp(\sigma \bar{\lambda}(\sigma))$  for all large  $\sigma$ ,
- (iv)  $\lim_{\sigma \rightarrow \infty} \frac{\log I_{\delta}(\sigma)}{\exp(\sigma \bar{\lambda}(\sigma))} = 1$ .

In this note we prove a theorem which gives us information as to how the functions  $\log I_{\delta}(\sigma)$  and  $\log J_{\delta,k}(\sigma)$  grow relative to each other as  $\sigma \rightarrow \infty$ . In what follows we shall prove the following:

**2. Theorem.** Let  $f(s)$  be an entire function represented by Dirichlet series, then

$$\begin{aligned} \lim_{\sigma \rightarrow \infty} \frac{\log I_{\delta}(\sigma)}{\log J_{\delta,k}(\sigma)} &\leq \bar{\lambda} \left\{ 1 + k \bar{\lambda} \right\} (1 + \bar{\lambda}/k), \quad (\bar{\lambda} > 0) \\ &\leq 1, \quad (\bar{\lambda} = 0). \end{aligned}$$

We need the following lemma for our purpose.

**Lemma.** For  $h > 0$ ,

$$\lim_{\sigma \rightarrow \infty} \frac{S(\sigma + h)}{S(\sigma)} = \exp(h\bar{\lambda})$$

where

$$S(\sigma) = \exp(\sigma \bar{\lambda}(\sigma)).$$

**Proof.** By a simple calculation, we have

$$\frac{S'(\sigma)}{S(\sigma)} = \sigma \bar{\lambda}'(\sigma) + \bar{\lambda}(\sigma).$$

Therefore, using the properties (i) and (ii), we see that for any  $\varepsilon > 0$ , there is a  $\sigma_0$  such that for every  $\sigma \geq \sigma_0$ ,

$$(\bar{\lambda} - \varepsilon) < S'(\sigma)/S(\sigma) < (\bar{\lambda} + \varepsilon).$$

Integrating the above inequality from  $\sigma$  to  $\sigma + h$ , we have

$$(\bar{\lambda} - \varepsilon)h < \log \left\{ \frac{S(\sigma + h)}{S(\sigma)} \right\} < (\bar{\lambda} + \varepsilon)h.$$

So that

$$\lim_{\sigma \rightarrow \infty} \frac{S(\sigma + h)}{S(\sigma)} = \exp(h\bar{\lambda}).$$

Now, we are in position to prove the theorem.

**Proof of Theorem.** We have

$$\begin{aligned} \log J_{\delta, k}(\sigma + h) &= \frac{1}{e^k(\sigma + h)} \int_0^{\sigma+h} \log I_{\delta}(x) e^{kx} dx, \\ &\geq \frac{1}{e^k(\sigma + h)} \int_{\sigma}^{\sigma+h} \log I_{\delta}(x) e^{kx} dx \\ &\geq \frac{\log I_{\delta}(\sigma)}{k e^{kh}} (e^{kh} - 1). \end{aligned}$$

So that, by the property (iv), we have

$$(2.1) \quad \begin{aligned} \lim_{\sigma \rightarrow \infty} \frac{\log J_{\delta,k}(\sigma)}{S(\sigma)} &\geq \lim_{\sigma \rightarrow \infty} \frac{\log I_{\delta}(\sigma)}{S(\sigma)} \left\{ \frac{e^{kh} - 1}{k e^{kh}} \right\} \\ &\geq \frac{e^{kh} - 1}{k e^{kh}} \end{aligned}$$

Put

$$\frac{\log J_{\delta,k}(\sigma + h)}{S(\sigma)} = \frac{\log J_{\delta,k}(\sigma + h)}{S(\sigma + h)} \frac{S(\sigma + h)}{S(\sigma)}$$

Here  $\log J_{\delta,k}(\sigma + h)/S(\sigma + h)$  and  $S(\sigma + h)/S(\sigma)$  are non-negative, so we have

$$\lim_{\sigma \rightarrow \infty} \frac{\log J_{\delta,k}(\sigma + h)}{S(\sigma)} \leq \lim_{\sigma \rightarrow \infty} \frac{\log J_{\delta,k}(\sigma + h)}{S(\sigma + h)} \exp(h\bar{\lambda})$$

by the lemma. This inequality with (2.1) will give us

$$\lim_{\sigma \rightarrow \infty} \frac{\log J_{\delta,k}(\sigma)}{S(\sigma)} \geq \frac{e^{kh} - 1}{k \exp((k + \bar{\lambda})h)}$$

Further, we have

$$(2.2) \quad \begin{aligned} \lim_{\sigma \rightarrow \infty} \frac{\log I_{\delta}(\sigma)}{\log J_{\delta,k}(\sigma)} &\leq \lim_{\sigma \rightarrow \infty} \left\{ \frac{\log I_{\delta}(\sigma)}{S(\sigma)} \right\} \\ &\quad \overline{\lim}_{\sigma \rightarrow \infty} \left\{ \frac{S(\sigma)}{\log J_{\delta,k}(\sigma)} \right\}, \\ &\leq 1 \cdot \frac{1}{\lim_{\sigma \rightarrow \infty} \frac{\log J_{\delta,k}(\sigma)}{S(\sigma)}} \\ &\leq \frac{k \exp((k + \bar{\lambda})h)}{e^{kh} - 1}, \quad (\bar{\lambda} > 0) \\ &\leq \frac{k \exp(kh)}{e^{kh} - 1} \quad (\bar{\lambda} = 0) \end{aligned}$$

Now, by usual method of calculus we minimize the right hand side of (2.2). We find that its minima is attained for that value of  $h$  which satisfies the relation

$$e^{kh} = \frac{k + \bar{\lambda}}{\bar{\lambda}} \quad \bar{\lambda} > 0$$

Substituting this value of  $h$  in (2.2), we get

$$\lim_{\sigma \rightarrow \infty} \frac{\log I_{\delta}(\sigma)}{\log J_{\delta,k}(\sigma)} \leq \bar{\lambda} [(1 + k/\bar{\lambda})]^{(1 + \bar{\lambda}/k)}$$

The case  $\bar{\lambda} = 0$ , is obvious.

This completes the proof of the theorem.

In conclusion, I wish to thank Dr. Shankar Hari Dwivedi, University of Udaipur and Dr. V. B. Goyal, University of Kurukshetra for useful discussions.

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#### ÖZET

Bu çalışmada,  $\log I_{\delta}(\alpha)$  ve  $\log J_{\delta,k}(\sigma)$  nin biri diğerine nazaran ne şekilde büyüdüklerini gösterir bir teorem ispatlanmıştır.

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